Extreme value statistics of 2d Gaussian Free Field: effect of finite domains

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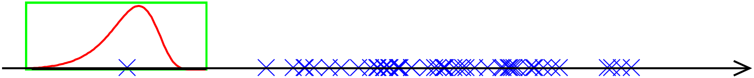
Acknowledgements: P. Le Doussal and Y. Fyodorov
J.Phys.A/Lett.(2016), cond.mat/1509.03187

Bristol, May 2016

Extreme-value statistics



$$\phi_{\min} := \min\{\phi_1, \ldots, \phi_M\} \xrightarrow{M\to\infty} a_M + b_M y$$

ϕ_i uncorrelated Gaussian	ϕ_i *log*-correlated Gaussian, *e.g.*,
$\overline{\phi_i \phi_j} = 2 \ln M \times \delta_{i,j}.$	$\overline{\phi_i \phi_j} = 2 \ln \lvert M/(i-j) \rvert$
$a_M = -2 \ln M + \frac{1}{2} \ln \ln M + O(1)$	$a_M = -2 \ln M + \frac{3}{2} \ln \ln M + O(1)$
$\mathrm{Pdf}(y \to -\infty) \sim e^{-\lvert y \rvert}$	$\mathrm{Pdf}(y \to -\infty) \sim \lvert y \rvert e^{-\lvert y \rvert}$

2d Gaussian Free Field

$\phi(z), z \in \mathbb{C}$ are Gaussian variables defined by

$$\overline{\phi(z)} = 0, \overline{\phi(z)\phi(w)} = -2 \ln |z - w|.$$

We study $y_M = \min_i \phi(z_i)$, $\{z_i\}_1^M$ being a mesh of a curve/region.

Only 2 solved cases:

- *circle*: $z_i = \exp(\mathbf{i}2\pi i/M)$ (Fyodorov-Bouchaud '08).
- *interval*: $z_j = j/M$ (Fyodorov-Le Doussal-Rosso '09).

Both solutions involve:

- Exact integrability;
- *Duality* of the solution;
- *Freezing* scenario.

UV *vs.* IR divergences

The planar Green function

$$\overline{\phi(z)\phi(w)} = -2 \ln |z - w| \tag{1}$$

has 2 divergences:

$z - w \to 0$ (UV):

- √ Gives rise to $\frac{3}{2}, |y| e^{-|y|}$.
- √ Its regularisation has no effect on y.

$z - w \to \infty$ (IR):

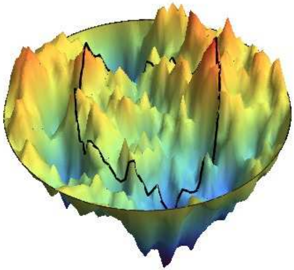
- ! Origin of **evil***.
- ⇒ Put GFF on a *finite* domain/surface;
- ⇒ Eq. (1) will be modified.

Goal questions: How is y affected? Duality/freezing?

* non-positivity of covariance matrix.

Our case study

Put ϕ on a unit disk (Dirichlet); consider the minima on a centred circle of radius $\sqrt{q}$ (Fyodorov-Bouchaud: $q \to 0$)



$$\overline{\phi_i \phi_j} = -2 \ln \frac{|1-z|}{|1-qz|}, \; z = e^{\mathbf{i}2\pi(i-j)/M} \neq 1, \; \overline{\phi_i^2} = 2 \ln M$$

Strategy: Thermodynamics of particle in random potential $\{\phi_i\}$

- The Partition function:

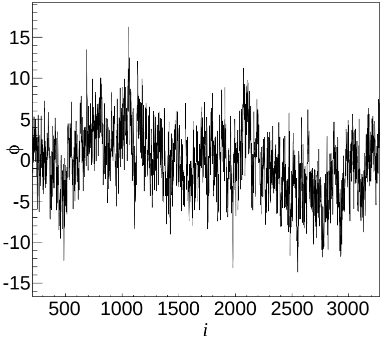
$$Z_M = \sum_{i=1}^{M} e^{-\beta\phi_i}, \beta = T^{-1}$$

- The free energy

$$F_M = -\frac{1}{\beta} \ln Z_M$$

gives $\phi_{\min} = \lim_{\beta\to\infty} F_M$.

! Phase transition at $\beta = 1$.



Strategy: high temperature phase $\beta < 1$

In general, we want to calculate replica averages

$$\overline{Z_M^n} = \sum_{i_1,\dots,i_n} \overline{\exp(-\beta(\phi_{i_1} + \dots + \phi_{i_n}))}$$

In high-T phase, one can approximate sum by integral:

$$\overline{Z_M^n} = \left(M e^{\frac{\beta^2}{2}\overline{\phi_i^2}}\right)^n \overline{Z^n}, \quad \overline{Z^n} = \int_0^{2\pi} \prod_{a=1}^{n} \frac{\mathrm{d}\theta_a}{2\pi} \prod_{a\neq b} e^{\frac{\beta^2}{2}\overline{\phi(\theta_a)\phi(\theta_b)}}$$

We shall calculate $\overline{Z^n}$ explicitly, and continue analytically the answer from $n = 1, 2, \dots$ to $n = -t/\beta \in \mathbb{C}$. This will give free energy fluctuation

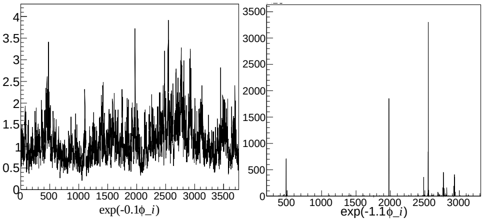
$$F_M = -(\beta + \beta^{-1}) \ln M + F, \quad \overline{\exp(tF)} = \overline{Z^{-t/\beta}}, \quad t \in \mathbb{C}.$$

Strategy: low temperature phase $\beta > 1$

When $\beta > 1$ ($T < 1$), the above is *wrong*, because

$$F_M = -(\beta + \beta^{-1}) \ln M + O(1) \Rightarrow S = \frac{\partial F_M}{\partial(1/\beta)} < 0 \text{ when } \beta > 1.$$

violates thermodynamics (entropy ≥ 0).
Reason: The sum Z_M cannot be approximated by integral.



Boltzmann weights for $\beta = 0.1$ and $\beta = 1.1$

Freezing and duality

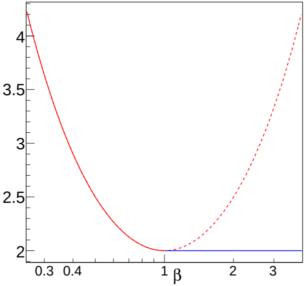
$$F_M = -(\beta + \beta^{-1}) \ln M + O(1)$$

implies that $S(\beta \nearrow 1) = 0$. Since $S \geq 0$ and $\partial_{\beta^{-1}} S \geq 0$,

$$S \overset{\beta>1}{=} 0 \Rightarrow F_M|_{\beta>1} = -2 \ln M + O(1)\,,$$

i.e., *freezing* of leading behaviour of free energy. What about fluctuations ?

Freezing of $\beta + \beta^{-1}$

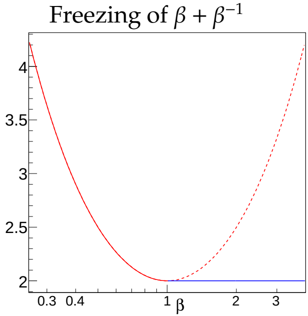


Freezing and duality conjecture

Observe in

$$F_M = \begin{cases} -(\beta + \beta^{-1}) \ln M + O(1) & \beta < 1 \\ -2 \ln M + O(1) & \beta > 1 \end{cases}$$

that $(\beta + \beta^{-1})$ is *dual* ($\beta \leftrightarrow 1/\beta$ invariant).
Conjecture: Dual quantities freeze, even for fluctuations.



Related to: KPP travelling wave, branching Brownian motion, 1 step replica symmetry breaking in spin glass ...

Details: Coulomb gas integral

Now we calculate and show duality of $\overline{Z^n}$:

$$\overline{Z^n} = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{n} \frac{\mathrm{d}\theta_i}{2\pi} \prod_{i<j} \left| \frac{z_i - z_j}{1 - q z_i \overline{z_j}} \right|^{-2\beta^2}, z_i := e^{\mathbf{i}\theta_i}$$

This Coulomb gas integral generalises

- the Dyson integral ($q = 0$),
- and Gaudin's models ($\beta^2 = -1$);

yet tractable thanks to *Jack polynomials*.

Jack polynomials

They are symmetric polynomials $J_\lambda(z = (z_1, \dots, z_n); \alpha)$ indexed by *partitions* λ.

We need only two properties:

- Cauchy identity

$$\prod_{i,j=1}^{n} (1 - q z_i w_j)^{-\alpha} = \sum_\lambda q^{|\lambda|} \frac{J_\lambda(z;\alpha) J_\lambda(w;\alpha)}{j_\lambda(\alpha)}$$

- Orthogonality:

$$\int_0^{2\pi} \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} \prod_{i<j} \left| z_i - z_j \right|^{2\alpha} \frac{J_\lambda(z;\alpha) J_\lambda(\bar{z};\alpha)}{j_\lambda(\alpha)}$$

$$= \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)^n} \prod_{(x,y)\in\lambda} \frac{x + \alpha(n-y)}{x + 1 + \alpha(n-y-1)}.$$

Jack polynomials

Apply Cauchy, then orthogonality

$$\overline{Z^n} = \int_0^{2\pi} \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} \frac{\prod_{i<j} \left|z_i - z_j\right|^{-2\beta^2}}{\prod_{i<j} \left|1 - q z_i \overline{z_j}\right|^{-2\beta^2}}$$

$$= \int_0^{2\pi} \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} \prod_{i<j} \left|z_i - z_j\right|^{-2\beta^2} \sum_{\lambda} \frac{J_\lambda(z; -\beta^2) J_\lambda(\overline{z}; -\beta^2)}{(1-q)^{n\beta^2} j_\lambda(-\beta^2)} q^{|\lambda|}$$

$$= \frac{\Gamma(1 - n\beta^2)}{\Gamma(1-\beta^2)^n (1-q)^{n\beta^2}} \sum_{\lambda} q^{|\lambda|} \prod_{(x,y)\in\lambda} \frac{x - \beta^2(n-y)}{x + 1 - \beta^2(n-y-1)}$$

Notice: the last formula makes sense for $n = -t/\beta \in \mathbb{C}$.

Result: free energy fluctuation ($\beta < 1$)

$$\overline{\exp(tF)} \overset{M\to\infty}{=} \left(\Gamma(1-\beta^2)^{\frac{1}{\beta}}(1-q)^{\beta}\right)^t \Gamma(1+t\beta)\mathbf{s}(t,\beta,q)$$

$\left(\Gamma(1-\beta^2)^{\frac{1}{\beta}}(1-q)^{\beta}\right)^t$ shifts $\overline{F}$, diverges as $\beta \to 1$ (related to $\frac{3}{2}\ln\ln$ term in $\beta > 1$ phase). So shift F to f such that

$$\overline{\exp(tf)} = \Gamma(1+t\beta)\mathbf{s}(t,\beta,q).$$

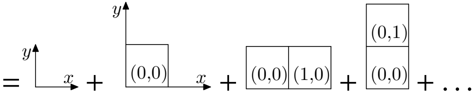
$\Gamma(1+t\beta)$ is Fyodorov-Bouchaud's solution for $q = 0$.
$\mathbf{s}(t,\beta,q)$ is new, encoding boundary effect:

$$\mathbf{s}(t,\beta,q) = \sum_{\lambda} \prod_{(x,y)\in\lambda} \frac{q(x\beta + y\beta^{-1} + t)}{(x+1)\beta + (y+1)\beta^{-1} + t}$$

Duality of s

We illustrate its first terms ($Q = \beta + \beta^{-1}$)

$$\mathbf{s} = \sum_{\lambda} \prod_{(x,y)\in\lambda} \frac{q(x\beta + y\beta^{-1} + t)}{(x+1)\beta + (y+1)\beta^{-1} + t}$$



$$= 1 + q\frac{t}{t+Q} + q^2 \frac{t}{(t+Q)(1+\frac{Q}{t+\beta})} + q^2 \frac{t}{(t+Q)(1+\frac{Q}{t+\beta^{-1}})} + \dots$$

It is **dual**: $\mathbf{s}(t,\beta,q) = \mathbf{s}(t,\beta^{-1},q)$. ($\beta \leftrightarrow \beta^{-1} \Leftrightarrow x \leftrightarrow y$)
This allows us to generalizes freezing/duality scenario from $q = 0$ to $q \neq 0$.

Freezing scenario

Although

$$\overline{\exp(tf)} = \Gamma(1+t\beta)\mathbf{s}(t,\beta,q)$$

is not dual, if we define $y_\beta = \beta^{-1}g + f$ where g is a standard Gumbel independent of f,

$$\overline{\exp(ty_\beta)} = \Gamma(1+t\beta^{-1})\Gamma(1+t\beta)\mathbf{s}(t,\beta,q)$$

is dual. So freezing scenario implies

$$\overline{\exp(ty_\beta)}|_{\beta>1} = \overline{\exp(ty_\beta)}|_{\beta=1} = \Gamma(1+t)^2\mathbf{s}(t,1,q)$$

Taking $\beta \to \infty$, we have

$$\lim_{\beta\to\infty} \overline{\exp(tf)} = \lim_{\beta\to\infty} \overline{\exp(ty_\beta)} = \Gamma(1+t)^2\mathbf{s}(t,1,q)$$

and obtain the minima distribution . . .

Numerical check of $\beta \le 1$ analytical continuation

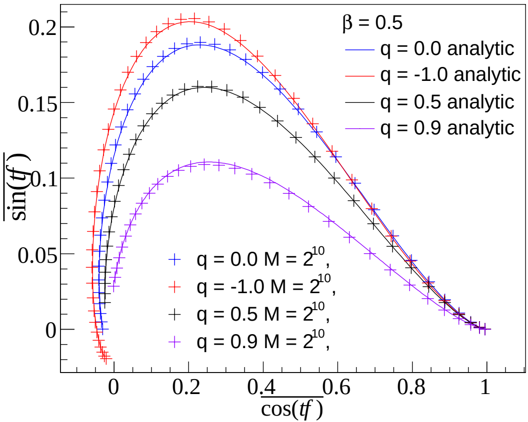


Figure : We test the prediction for $\overline{\exp(\mathbf{i}tf)}$ in the $\beta < 1$ phase. The first moment is subtracted from both numerics and analytic prediction.

Minimum distribution

Since the high-T result is valid and *duality* satisfied, we apply freezing scenario to obtain minima distribution:

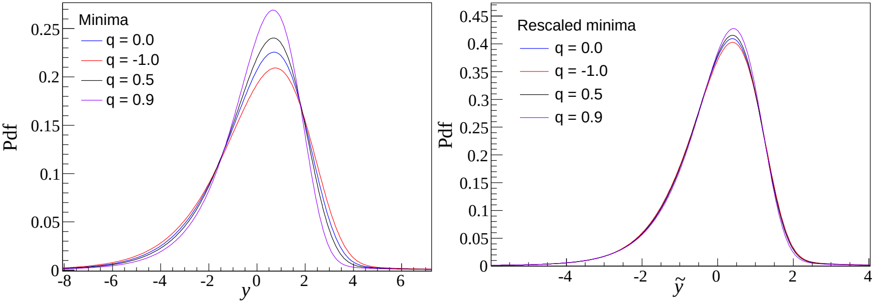


Figure : Left: The mean is shifted to zero. Right: The distributions are rescaled to have zero mean and unit variance.

Minimum distribution: Carpentier-Le Doussal tail

From the pole structure of

$$\overline{\exp(tf)}|_{\beta\to\infty} = \Gamma(1+t)^2 \sum_{\lambda} \prod_{(x,y)\in\lambda} \frac{q(x+y+t)}{2+x+y+t}$$

we deduce that the tail behaviour of $f \to -\infty$ is given by the rightmost double pole of $\Gamma(1+t)^2$ at $t = -1$. In the sum, only $\lambda = \emptyset$ and $\lambda = \square = \{(0,0)\}$ contribute, so

$$\mathrm{Pdf}(f \to -\infty) \sim (1-q)|f|e^{-|f|}.$$

Conclusion

We studied the minima of GFF on a circle in a disk with Dirichlet boundary condition:

- Minimum distribution is non-trivially affected by the presence of boundary.
- Duality and freezing scenario still hold.
- Technical contributions: an exact formula for a Coulomb-gas integral, . . .

Perspective: understand how to proceed *without* integrability . . .

Numerical test of freezing

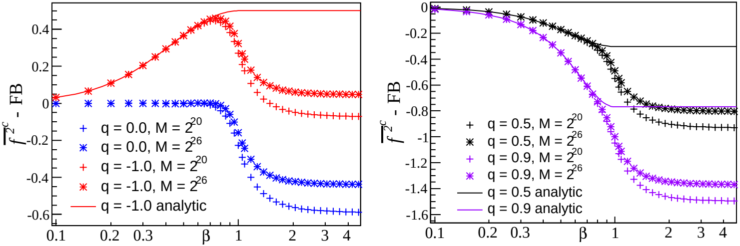


Figure : The variance of $\overline{f^2}^c$ with the FB ($q = 0$) analytic values subtracted. The analytic-numerics agreement is excellent in the high-T phase and degrades when approaching the transition. In the low temperature phase, the plateau indicating freezing is observed for all numerical data.

Fine-tuning the definition

$$\overline{\phi_i \phi_j} = -2 \ln \frac{|1-z|}{|1-qz|} + a, z = e^{\mathbf{i}2\pi(i-j)/M}$$

- $a = -\ln q$ is the *zero mode*. Its effect on y is trivial. We will discard it.
- In terms of Fourier modes

$$\phi_j = \Re \sum_{k \neq 0} \sqrt{\mu_k} \exp(\mathbf{i}2\pi jk/M), \mu_k = \frac{(1-q^{|k|})}{|k|}$$

UV regularisation: $\sum_{k\neq 0} \to \sum_{k=-M/2}^{M/2-1}$.