fBM with zero Hurst index and the GUE

Boris Khoruzhenko

Queen Mary University of London

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A. Motivation

Log-correlated random fields and RMT

Log-correlated fields emerge in the large-N limit of $D_N = -\log|\det(\mathcal{H} - xI)|$, where $\mathcal{H}$ is $N \times N$ random matrix. Earliest examples

Hughes, Keating & O'Connell 2001 (periodic $1/f$-noise, CUE)

Rider & Virag 2007 (Gaussian Free Field, Ginibre Ensemble)

Our work is extension of HKO'C01 to mesoscopic scales in the spectrum.

Consider the simplest RM ensembles...

B. Glossary: Matrix Ensembles

CUE: Haar unitary matrices, egvs on the unit circle, e^{ix_j}, periodic

GUE: Gaussian Hermitian, $p(\mathcal{H}) \propto e^{-2N\,\mathrm{Tr}\,\mathcal{H}^2}$, real egvs x_j, boundaries

Induced JPDF of $x_1, \ldots x_N$:

$$P_{CUE}^{(N)}(x_1, \ldots x_N) \propto \prod_{j<k} |e^{ix_j} - e^{ix_k}|^2,\ P_{GUE}^{(N)}(x_1, \ldots x_N) \propto e^{-2N\sum x_j^2} \prod_{j<k} |x_j - x_k|^2$$

From this one derives marginal distributions for $x_1, \ldots x_n, n < N$

$$R_n^{(N)}(x_1, \ldots x_n) \propto \det\left[K_{ensemble}^{(N)}(x_j, x_k)\right]_{j,k=1,\ldots,n}$$

In particular, for the density of egvs averaged on ensemble distribution:

$$CUE \quad R_1^{(N)}(x) = \frac{N}{2\pi} \quad \text{and} \quad GUE \quad R_1^{(N)}(x) \approx \frac{2N}{\pi}\sqrt{1-x^2}, N \gg 1.$$

Have boundaries at $x = \pm 1$ in GUE and no boundaries in CUE.

B. Glossary: Spectral scales

Fix x_0 in the bulk of the spectrum, consider the interval $I = (x_0, x_0 + s/d_N)$.

- Macroscopic scale ($d_N = 1$): av. no. of egvs in I, $\int_{x_0}^{x_0+s/d_N} R_1^{(N)}(x)dx \propto N$,
- Microscopic scale ($d_N = N$): av. no. of egvs in I is finite N
- Mesoscopic scale ($1 \ll d_N \ll N$, e.g. $d_N = N^\alpha, \alpha \in (0,1)$)

D_N is an example of a linear statistic $X[f] = \sum_j f(d_N(x_k - x_0))$, test fnc f.

For CUE the asympt distr of $X[f]$ is easy to work out on the macro scale.

$$\mathbf{E}\left\{e^{\sum_j f(x_j)}\right\} = \det\left(\int_0^{2\pi} e^{f(x)} e^{i(j-k)x} dx\right) \approx e^{N\hat{f}(0) + \frac{1}{2}\sum_k |k||\hat{f}(k)|^2}$$

by strong Szegö theorem for Töplitz dets. Thus, $X[f] - \mathbf{E}\{X[f]\}$ is asympt normal provided the variance is finite and minor reg conditions on test fnc.

C. Log-characteristic polynomials: macro scale

Consider

$$\begin{aligned} D_N(x) &= -\log|\det(I - e^{-ix}U)|, \quad U \in \mathrm{CUE}, x \in (0, 2\pi); \\ &= -\log|\det(xI - \mathcal{H})|, \qquad \mathcal{H} \in \mathrm{GUE}, x \in (-1, 1) \end{aligned} \tag{1}$$

Test function fails the finite variance condition, $\mathrm{Var}\{D_N\}$ grows with N. Normalising by st dev delivers asympt normality. But correlations are in next-to-leading order.

HKO'C01: don't normalise and consider $D_N(x)$ as a generalised random function, establish convergence in a suitable fnc space.

For CUE, can work out the limiting (Gaussian) process by Taylor expansion

$$-\log|\det(I - e^{-ix}U)| = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(v_n e^{-inx} + c.c.), \quad v_n = \frac{1}{\sqrt{n}} \mathrm{Tr}\, U^n.$$

By Diaconis-Shahshahani 1994, $(v_1, \ldots, v_n)$ are asympt normal, $v_1, v_2, \ldots$ are i.i.d. CN(0,1). Then $D_N(x)$ converges to log-correlated Gaussian process

$$F(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(v_n e^{-inx} + c.c.), \quad \mathbf{E}\{F(x)F(y)\} = -\frac{1}{2}\log\left|1 - e^{i(x-y)}\right|$$

C. Log-characteristic polynomials: macroscopic scale

Similar argument holds for GUE. Have Haagerup identity : for $x \neq y$

$$-\log(2|x-y|) = \sum_{n=1}^{\infty} \frac{2}{n} T_n(x)T_n(y), \qquad x, y \in [-1,1]$$

$$= -\log|x+\sqrt{x^2-1}| + \sum_{n=1}^{\infty} \frac{2}{n}(x-\sqrt{x^2-1})^n T_n(y) \quad x > 1, |y| \leq 1.$$

where $T_n(x) = \cos(n \arccos x)$ are Chebyshev polynomials. From this

$$-\log|\det[xI - \mathcal{H}] = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} T_n(x) + N \log 2 + R_N(x),$$

where $a_n = \frac{2}{\sqrt{n}} \operatorname{Tr} T_n(\mathcal{H})$ and the term R_N collects contributions from egvs of H outside $[-1,1]$.

Johansson 1998 proved that $(a_1, \ldots, a_m) \Rightarrow$ i.i.d. N(0,1). Then $D_N(x)$ converges to log-correlated Gaussian process

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} T_n(x), \quad \mathbf{E}\{F(x)F(y)\} = -\frac{1}{2}\log|2(x-y)|$$

C. Log-characteristic polynomials: macroscopic scale

The Chebyshev-Fourier expansion approach can be made rigorous, more or less same as in HKO'C01 except that now have to deal with contributions from the tails of egv distribution.

Define Sobolev space

$$V^{(a)} = \{f \in L^2((-1,1), \frac{dx}{\sqrt{1-x^2}}) : \sum_{k=1}^{\infty}(1+k^2)^a |c_k(f)|^2 < \infty\}, \quad a > 0,$$

where $c_k(f)$ are Chebyshev-Fourier coefficients. $V^{(-a)}$ is the dual to $V^{(a)}$.

With prob 1, $F(x) = \sum_{k=1}^{\infty} \frac{a_k}{\sqrt{k}} T_k(x) \in V^{(-a)}$ for all $a > 0$; $F[f] = \sum_{k=1}^{\infty} \frac{a_k}{\sqrt{k}} c_k(f)$.

$F(x)$ is zero mean Gaussian generalised process with covariance

$$\mathbf{E}\{F[f]F[g]\} = \int\int \Big(-\frac{1}{2}\log 2|x-y|\Big) f(x)f(y)dxdy.$$

Thm. In the limit $N \to \infty$, $D_N(x) - \mathbf{E}\{D_N(x)\} \Rightarrow F(x)$ in $V^{(-a)}$ for any $a > 1/2$.

D. Fractional Brownian Motion

fBM: Gaussian self-similar process with stationary increments

$$B_H(0) = 0, \quad \mathbf{E}\{B_H(t)\} = 0, \quad \mathrm{Var}\{|B_H(t) - B_H(s)|^2\} = \sigma^2 |t-s|^{2H}$$

Have $\{B_H(at), t \geq 0\} = |a|^H \{B_H(t), t \geq 0\}$ in distribution on cylinder sets.

The scaling exponent H is called the Hurst index (British hydrologist).

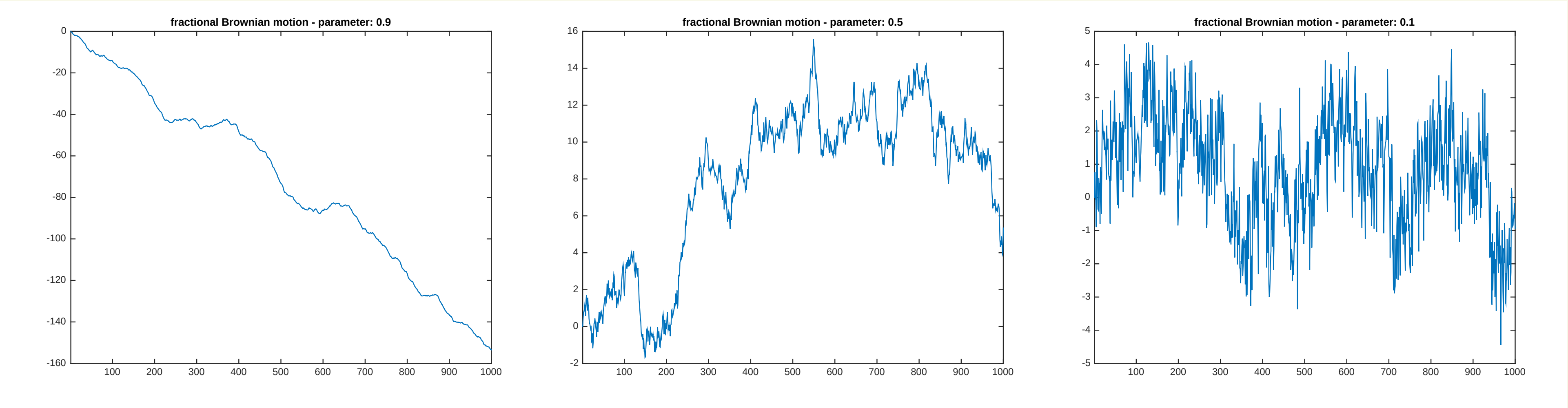
Successive increments of fBM are

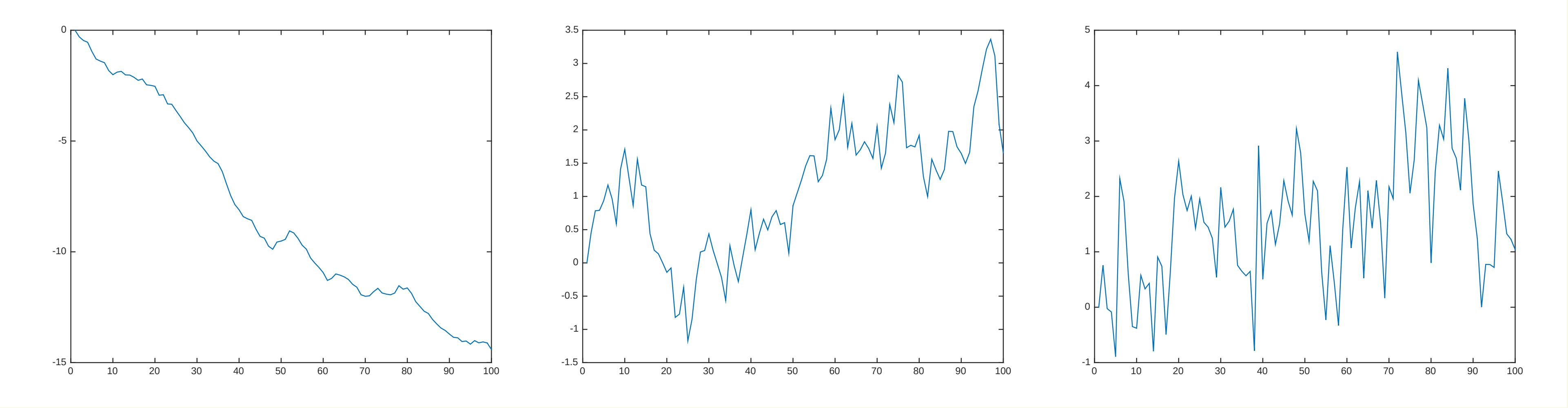
- uncorrelated if $H = 1/2$, have usual Brownian Motion
- positively correlated if $1/2 < H < 1$, and
- negatively correlated if $0 < H < 1/2$ ('zigzagy' paths)

Sample paths are Hölder continuous with $\lambda < H$.

Hausdorff dimension of $(t, B_H(t))_{t \in [0,1]} = 2 - H$, for H small the trajectory of fBM is space filling.

D. Fractional Brownian Motion





D. Regularised Fractional Brownian Motion with H=0

Have stochastic Fourier integral representation for fBM

$$B_H(t) = \frac{1}{2\sqrt{2}} \int_0^\infty \frac{1}{s^{H+1/2}} \left[(e^{-its} - 1) dB_c(s) + c.c. \right] ,$$

where $B_c(s) = B_{1/2}(s) + iB_{1/2}(s)$ complex Brownian Motion.
Define

$$B^{(\eta)}{}_H(t) = \frac{1}{2\sqrt{2}} \int_0^\infty \frac{e^{-\eta s}}{s^{H+1/2}} \left[(e^{-its} - 1) dB_c(s) + c.c. \right] .$$

$B_H^{(\eta)}(t)$ is a zero mean Gaussian process with stationary increments,

$$\mathrm{Var}\{|B_H^{(\eta)}(t) - B_H^{(\eta)}(s)|^2\} = \int_0^\infty \frac{e^{-2\eta s}}{s^{1+2H}} (1 - \cos(ts)) ds = 2\phi_H^{(\eta)}(t-s)$$

In the limit $\eta \to 0$, $\phi_H^{(\eta)}(t) \to \frac{1}{4H}\Gamma(1-2H)\cos(\pi H)|t|^{2H}$ if $H > 0$, recover fBM!

For fixed $\eta > 0$, $\lim_{H \to 0} \phi_H^{(\eta)}(t) = \frac{1}{4} \log\left(1 + \frac{t^2}{4\eta^2}\right)$

Regularised fBM with zero Hurst index, $B_0^{(\eta)}(t)$,

$$B_0^{(\eta)}(0) = 0, \quad \mathbf{E}\{B_0^{(\eta)}(t)\} = 0, \quad \mathrm{Var}\{|B_0^{(\eta)}(t) - B_0^{(\eta)}(s)|^2\} = \frac{1}{4}\log\left(1 + \frac{t^2}{4\eta^2}\right)$$

E. Log-characteristic polynomials: mesoscopic scale

Previously looked at $D_N(x) = -\log|\det[x - \mathcal{H}]|$ on the macro scale.

Now, 'zoom in' on a point x_0 in the bulk, $D_N(\tau) = -\log\left|\det\left(x_0 + \frac{t}{d_N} - \mathcal{H}\right)\right|$

Regularise ($\eta > 0$):

$$W_N^\eta(t) = -\log\left|\det\left(x_0 + \frac{t + i\eta}{d_N} - \mathcal{H}\right)\right| + \log\left|\det\left(x_0 + \frac{i\eta}{d_N} - \mathcal{H}\right)\right|$$

Revisiting series expansions... By making use of an exact identity, can write

$$W_N^\eta(t) = \frac{1}{2}\int_0^\infty \frac{e^{-\eta s}}{\sqrt{s}}\left[(e^{-its} - 1)b_N(s)ds + c.c.\right], \quad b_N(s) = \frac{1}{\sqrt{s}}\,\mathrm{Tr}\, e^{isd_N(\mathcal{H}-x)}$$

Heuristic arguments suggest $b_N(s) \to dB_c(t)$ as $N \to \infty$ om mesoscopic scales (analogue of Diaconis-Shahshahani), but we were unable to push through this approach.

Linear egvs stats on mesoscopic scale: Boutet de Monvel & Khorunzhy 1999 (GUE, Wigner, trace of resolvent), Soshnikov 2000 (CUE, smooth test fncs).

E. Log-characteristic polynomials: mesoscopic scale

Thm $W_N^\eta(t) - \mathbf{E}\{W_N^\eta(t)\}$ converges to $B_0^{(\eta)}(t)$ on cylinder sets as $N \to \infty$, provided that

$$d_N \to \infty \text{ and } d_N = o(N/\log N).$$

Also have weak convergence in $L^2(a,b)$ on finite intervals.

Proof by the R-H problem machinery. Write the characteristic fnc as Hankel det,

$$\mathbf{E}\Big\{e^{\sum_{k=1}^{m+1}\alpha_k W_N^\eta(t_k)}\Big\} = \det\left(\int_R x^{k+j} w_N(x)dx\right)_{k,j=0}^{m}, \quad w_N(x) = e^{-Nx^2}\prod_{k=1}^{m+1}|x - z_k|,$$

where $z_k = x_0 + (t_k + i\eta)/d_N$, and then follow Krasovsky 2006. In our case, the singularities of weight function merge at x_0 in the limit $N \to \infty$.

This result was recently extended by Lohdia & Simm 2015, and He & Knowles 2016 to Wigner matrices, and by Lambert 2016 to (single cut) unitary invariant ensembles. Range of mesoscopic scales $d_N = N^\alpha, 0 < \alpha < 1$.

E. Diaconis-Shahshahani on mesoscopic scale

Recall

$$W_N^\eta(t) = \frac{1}{2}\int_0^\infty \frac{e^{-\eta s}}{\sqrt{s}} \left[(e^{-its} - 1)b_N(s)ds + c.c.\right], \quad b_N(s) = \frac{1}{\sqrt{s}} \operatorname{Tr} e^{isd_N(\mathcal{H}-x)}$$

Consider $c_N(\xi) = \int \xi(s) b_N(s) ds$ for complex-valued test functions ξ.

Thm Suppose $d_N = N^\alpha, 0 < \alpha < 1$. For any finite number of $\xi_j \in C_0^\infty(\mathbf{R}_+)$

$(c_N(\xi_1), \ldots, c_N(\xi_m))$ converges to a zero mean Gaussian vector $\mathbf{Z}$ having zero relation matrix, $\mathbf{E}\{\mathbf{Z}\mathbf{Z}^T\} = 0$ and covariance

$$\mathbf{E}\{\mathbf{Z}\overline{\mathbf{Z}}^T\} = (\Gamma_{jk}), \quad \Gamma_{jk} = \int \xi_j(s)\overline{\xi}_k(s) ds$$

This can be thought of as a continuous generalisation of Diaconis-Shahshahani.

THANK YOU