Extreme values of Riemann zeta over $(0, T)$

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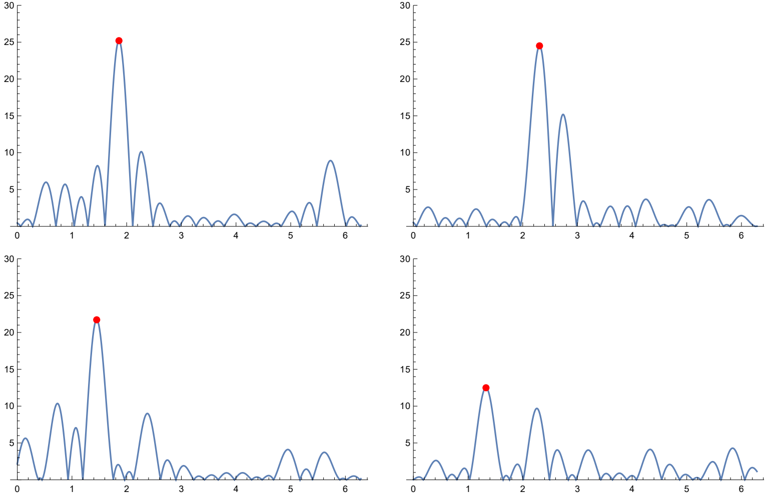
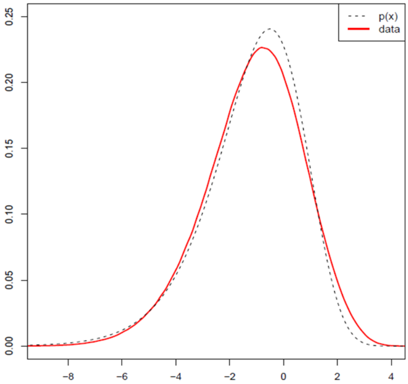
Extreme values of Riemann zeta in finite intervals



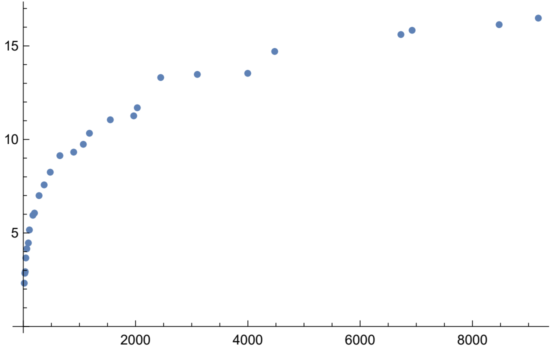
The largest value of zeta over an interval of length 2π
for $t = 10^{10}, 10^{10} + 100, 10^{10} + 200, 10^{10} + 300$

Extreme values of Riemann zeta in finite intervals



Distribution of $-2 \log \max_{t \in [T, T+2\pi]} |\zeta(\frac{1}{2} + \mathrm{i}t)|$ (after rescaling to get the empirical variance to agree) based on 2.5×10^8 zeros near $T = 10^{28}$. Graph by Ghaith Hiary, taken from Fyodorov-Keating.

Extreme values of Riemann zeta over $(0, T)$



The running maxima of zeta
for $0 \leq t \leq 10^3$

Extreme values of zeta

Conjecture (Farmer, Gonek, Hughes)

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log T \log\log T}\right)$$

Bounds on extreme values of zeta

Theorem (Littlewood; Ramachandra and Sankaranarayanan; Soundararajan; Chandee and Soundararajan)

Under RH, there exists a C such that

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| = O\left(\exp\left(C\frac{\log T}{\log\log T}\right)\right)$$

Theorem (Bondarenko-Seip)

For all $c < 1/\sqrt{2}$

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2}+\mathrm{i}t)| > \exp\left(c\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\right)$$

An Euler-Hadamard hybrid

Theorem (Gonek, Hughes, Keating)

A simplified form of our theorem is:

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t; X) Z(t; X) + \textit{errors}$$

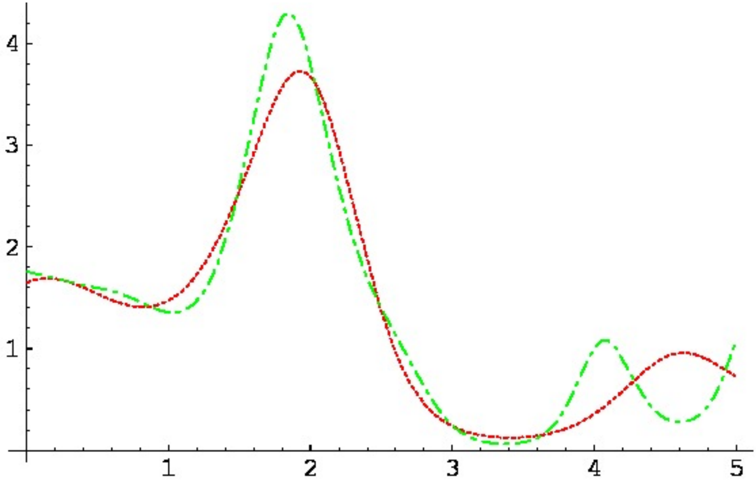
where

$$P(t; X) = \prod_{p \le X} \left(1 - \frac{1}{p^{\frac{1}{2} + \mathrm{i}t}}\right)^{-1}$$

and

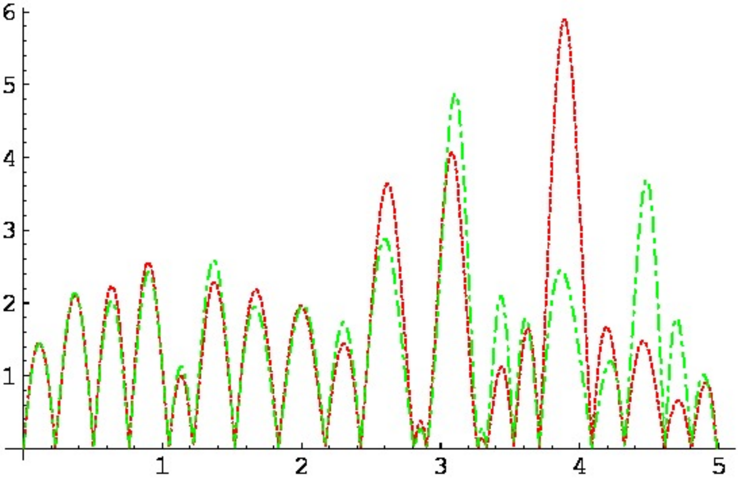
$$Z(t; X) = \exp\left(\sum_{\gamma_n} \mathrm{Ci}(|t - \gamma_n| \log X)\right)$$

An Euler-Hadamard hybrid: Primes only



Graph of $|P(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$,
with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

An Euler-Hadamard hybrid: Zeros only



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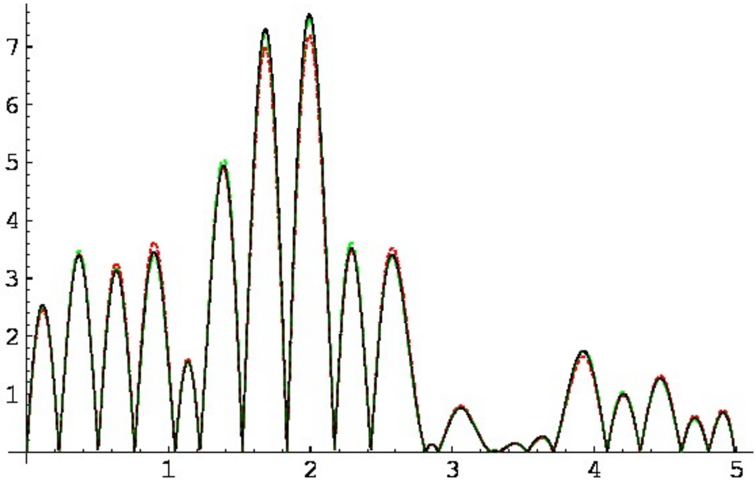
An Euler-Hadamard hybrid: Primes and zeros



Graph of $|\zeta(\frac{1}{2} + i(t + t_0))|$ (black) and $|P(t + t_0; X)Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).

Modeling zeta with RMT

Keating and Snaith showed that the Riemann zeta function can be modelled with characteristic polynomials of random unitary matrices with size

$$N = \log \frac{T}{2\pi}$$

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Similarly $Z(t; X)$ can be modelled with random matrix theory, and the results look like characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma} \log X}$$

Extreme values of zeta: RMT & zeros

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blocks, each containing approximately N zeros.

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Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.
Find the smallest $K = K(M, N)$ such that choosing M independent characteristic polynomials of size N, almost certainly none of them will be bigger than K.

Extreme values of zeta: RMT & zeros

Note that

$$\mathbb{P}_{U(N)}\left\{\max_{1\le j\le M}\max_{\theta}\left|\Lambda_{A_j}(e^{i\theta})\right|\le K\right\}=\mathbb{P}_{U(N)}\left\{\max_{\theta}\left|\Lambda_{A}(e^{i\theta})\right|\le K\right\}^M$$

Theorem

Let $0<\beta<2$*. If* $M=\exp(N^\beta)$*, and if*

$$K=\exp\left(\sqrt{\left(1-\tfrac{1}{2}\beta+\varepsilon\right)\log M\log N}\right)$$

then

$$\mathbb{P}_{U(N)}\left\{\max_{1\le j\le M}\max_{\theta}\left|\Lambda_{A_j}(e^{i\theta})\right|\le K\right\}\to 1$$

as $N\to\infty$ *for all* $\varepsilon>0$*, but for no* $\varepsilon<0$*.*

Extreme values of zeta: RMT & zeros

Recall

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t;X)Z(t;X) + \text{errors}$$

We showed that $Z(t;X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^\gamma \log X}$$

Recall

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t;X)Z(t;X) + \text{errors}$$

We showed that $Z(t;X)$ can be modelled by characteristic polynomials of size

$$N = \frac{\log T}{e^{\gamma} \log X}$$

Therefore the previous theorem suggests

Conjecture

If $X = \log T$*, then*

$$\max_{t \in [0,T]} |Z(t;X)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log T \log\log T} \right)$$

Extreme values of zeta: RMT & zeros

Theorem

By the PNT, if $X = \log T$ *then for any* $t \in [0, T]$*,*

$$P(t; X) = O\left(\exp\left(C\frac{\sqrt{\log T}}{\log\log T}\right)\right)$$

Thus one is led to the max values conjecture

Conjecture

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2} + \mathrm{i}t)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right)$$

Extreme values of zeta: Primes

First note that

$$P(t;X) = \exp\left(\sum_{p\le X} \frac{1}{p^{1/2+\mathrm{i}t}}\right) \times O(\log X)$$

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Treat $p^{-\mathrm{i}t}$ as independent random variables, U_p, distributed uniformly on the unit circle.
The distribution of

$$\mathfrak{Re} \sum_{p\le X} \frac{U_p}{\sqrt{p}}$$

tends to Gaussian with mean 0 and variance $\frac{1}{2} \log\log X$ as $X \to \infty$.

Extreme values of zeta: Primes

We let $X = \exp(\sqrt{\log T})$ and model the maximum of $P(t;X)$ by finding the maximum of the Gaussian random variable sampled $T(\log T)^{1/2}$ times. This suggests

$$\max_{t\in[0,T]} |P(t;X)| = O\left(\exp\left((\frac{1}{\sqrt{2}}+\varepsilon)\sqrt{\log T \log\log T}\right)\right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$.

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for all $\varepsilon > 0$ and no $\varepsilon < 0$.
For such a large X, random matrix theory suggests that

$$\max_{t\in[0,T]} |Z(t;X)| = O\left(\exp\left(\sqrt{\log T}\right)\right).$$

This gives another justification of the large values conjecture.

Other arguments

Arguments for:

- Values of X between $\log T$ and $\exp(\sqrt{\log T})$, when both $Z(t;X)$ and $P(t;X)$ are large.
- High moments of zeta

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- Values of X between $\log T$ and $\exp(\sqrt{\log T})$, when both $Z(t;X)$ and $P(t;X)$ are large.
- High moments of zeta

Arguments against:

- Makes use of independence between blocks of size $O(1)$
- Actual extreme values might be too rare to be caught by this probabilistic method
- It's number theory! (Think about $d(n) = 2^{\omega(n)}$)

Bibliography

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