Extreme values of Riemann zeta over \((0, T)\)

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The largest value of zeta over an interval of length $2\pi$ for $t = 10^{10}, 10^{10} + 100, 10^{10} + 200, 10^{10} + 300$. 

Extreme values of Riemann zeta in finite intervals.
Distribution of $-2 \log \max_{t \in [T, T+2\pi]} |\zeta(\frac{1}{2} + it)|$ (after rescaling to get the empirical variance to agree) based on $2.5 \times 10^8$ zeros near $T = 10^{28}$. Graph by Ghaith Hiary, taken from Fyodorov-Keating.
Extreme values of Riemann zeta over $(0, T)$

The running maxima of zeta for $0 \leq t \leq 10^3$
Conjecture (Farmer, Gonek, Hughes)

\[
\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp \left( \left( \frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log T \log \log T} \right)
\]
Bounds on extreme values of zeta

Theorem (Littlewood; Ramachandra and Sankaranarayanan; Soundararajan; Chandee and Soundararajan)

Under RH, there exists a $C$ such that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = O \left( \exp \left( C \frac{\log T}{\log \log T} \right) \right)$$

Theorem (Bondarenko-Seip)

For all $c < 1/\sqrt{2}$

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| > \exp \left( c \sqrt{\frac{\log T \log \log \log \log T}{\log \log T}} \right)$$
An Euler-Hadamard hybrid

Theorem (Gonek, Hughes, Keating)

A simplified form of our theorem is:

\[ \zeta \left( \frac{1}{2} + it \right) = P(t; X)Z(t; X) + \text{errors} \]

where

\[ P(t; X) = \prod_{p \leq X} \left( 1 - \frac{1}{p^{2+it}} \right)^{-1} \]

and

\[ Z(t; X) = \exp \left( \sum_{\gamma_n} \text{Ci}(|t - \gamma_n| \log X) \right) \]
An Euler-Hadamard hybrid: Primes only

Graph of $|P(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).
An Euler-Hadamard hybrid: Zeros only

Graph of $|Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12} + 40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).
Graph of $|\zeta\left(\frac{1}{2} + i(t + t_0)\right)|$ (black) and $|P(t + t_0; X)Z(t + t_0; X)|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0 \approx 26$ (red) and $X = 1000$ (green).
Keating and Snaith showed that the Riemann zeta function can be modelled with characteristic polynomials of random unitary matrices with size

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\[ N = \log \left( \frac{T}{2\pi} \right) \]

Similarly \( Z(t; X) \) can be modelled with random matrix theory, and the results look like characteristic polynomials of size

\[ N = \frac{\log T}{e^\gamma \log X} \]
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blocks, each containing approximately $N$ zeros.
Model each block with the characteristic polynomial of an $N \times N$ random unitary matrix.
Find the smallest $K = K(M, N)$ such that choosing $M$ independent characteristic polynomials of size $N$, almost certainly none of them will be bigger than $K$. 
Note that

\[ \mathbb{P}_{U(N)} \left\{ \max_{1 \leq j \leq M} \max_\theta \left| \Lambda_j(e^{i\theta}) \right| \leq K \right\} = \mathbb{P}_{U(N)} \left\{ \max_\theta \left| \Lambda(e^{i\theta}) \right| \leq K \right\} \]

**Theorem**

Let \( 0 < \beta < 2 \). If \( M = \exp(N^\beta) \), and if

\[ K = \exp \left( \sqrt{(1 - \frac{1}{2}\beta + \varepsilon) \log M \log N} \right) \]

then

\[ \mathbb{P}_{U(N)} \left\{ \max_{1 \leq j \leq M} \max_\theta \left| \Lambda_j(e^{i\theta}) \right| \leq K \right\} \to 1 \]

as \( N \to \infty \) for all \( \varepsilon > 0 \), but for no \( \varepsilon < 0 \).
Recall
\[ \zeta\left(\frac{1}{2} + it\right) = P(t; X)Z(t; X) + \text{errors} \]

We showed that \( Z(t; X) \) can be modelled by characteristic polynomials of size
\[ N = \frac{\log T}{e^{\gamma} \log X} \]
Recall

\[ \zeta(\frac{1}{2} + it) = P(t; X)Z(t; X) + \text{errors} \]

We showed that \( Z(t; X) \) can be modelled by characteristic polynomials of size

\[ N = \frac{\log T}{e^\gamma \log X} \]

Therefore the previous theorem suggests

**Conjecture**

If \( X = \log T \), then

\[ \max_{t \in [0, T]} |Z(t; X)| = \exp \left( \left( \frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log T \log \log T} \right) \]
Theorem

By the PNT, if $X = \log T$ then for any $t \in [0, T]$,

$$P(t; X) = O\left(\exp\left(C \frac{\sqrt{\log T}}{\log \log T}\right)\right)$$

Thus one is led to the max values conjecture

Conjecture

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log T \log \log T}\right)$$
First note that

\[ P(t; X) = \exp \left( \sum_{p \leq X} \frac{1}{p^{1/2 + it}} \right) \times O(\log X) \]
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Treat \( p^{-it} \) as independent random variables, \( U_p \), distributed uniformly on the unit circle. The distribution of

\[ \Re \sum_{p \leq X} \frac{U_p}{\sqrt{p}} \]

tends to Gaussian with mean 0 and variance \( \frac{1}{2} \log \log X \) as \( X \to \infty \).
We let $X = \exp(\sqrt{\log T})$ and model the maximum of $P(t; X)$ by finding the maximum of the Gaussian random variable sampled $T(\log T)^{1/2}$ times. This suggests

$$\max_{t \in [0, T]} |P(t; X)| = O \left( \exp \left( \left( \frac{1}{\sqrt{2}} + \varepsilon \right) \sqrt{\log T \log \log T} \right) \right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$. 
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$$\max_{t \in [0, T]} |P(t; X)| = O\left(\exp\left((\frac{1}{\sqrt{2}} + \varepsilon)\sqrt{\log T \log \log T}\right)\right)$$

for all $\varepsilon > 0$ and no $\varepsilon < 0$.

For such a large $X$, random matrix theory suggests that

$$\max_{t \in [0, T]} |Z(t; X)| = O\left(\exp\left(\sqrt{\log T}\right)\right).$$

This gives another justification of the large values conjecture.
Other arguments

Arguments for:

- Values of $X$ between $\log T$ and $\exp(\sqrt{\log T})$, when both $Z(t; X)$ and $P(t; X)$ are large.
- High moments of zeta
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- Values of $X$ between $\log T$ and $\exp(\sqrt{\log T})$, when both $Z(t; X)$ and $P(t; X)$ are large.
- High moments of zeta

Arguments against:
- Makes use of independence between blocks of size $O(1)$
- Actual extreme values might be too rare to be caught by this probabilistic method
- It's number theory! (Think about $d(n) = 2^{\omega(n)}$)

