Moments of β-ensembles

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β-ensembles

- Eigenvalue joint probability density function:

$$P\left(\vec{\lambda}\right) = \frac{1}{\mathcal{N}} \prod_{i=1}^{N} w\left(\lambda_i\right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^{\beta}$$

  - Gaussian - $w(\lambda) = \exp\left(-\frac{\lambda^2}{2}\right)$; $\lambda \in \mathbb{R}$
    - GOE : $\beta = 1$
    - GUE : $\beta = 2$
  - Laguerre - $w(\lambda) = \lambda^{\gamma} \exp\left(-\lambda\right)$; $\lambda \geq 0$
  - Jacobi - $w(\lambda) = \lambda^{\gamma_1} \left(1 - \lambda\right)^{\gamma_2}$; $-1 \leq \lambda \leq 1$
- $\beta > 0$ as a continuous parameter
  - Log-gas
  - Tridiagonal realisation
  - Calogero-Sutherland models

Moments

- Let

$$\left\langle f\left(\vec{\lambda}\right)\right\rangle = \int_{\mathcal{D}} f\left(\vec{\lambda}\right) P\left(\vec{\lambda}\right) \mathrm{d}\lambda_1 ... \mathrm{d}\lambda_N$$

- The moments are:

$$\mu_k = \left\langle \sum_{j=1}^{N} \lambda_j^k \right\rangle$$

- Why are moments of β-ensembles interesting?
  - The p.d.f. of the position of the maximum of certain log-correlated processes on the interval (including the logarithm of the GUE characteristic polynomial) can be studied through it's moments - which requires calculating the moments of the JβE [Fyodorov/Le Doussal 2015].
  - Genus expansions (GβE)

$$\frac{1}{N}\mu_{2k} = \sum_{g=0}^{[k/2]} \frac{c(g;k)}{N^{2g}}$$

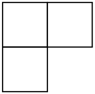
The Result

- Our formula for the moments in terms of the averages of Jack polynomials is given by:

$$\mu_k = \sum_{\nu \vdash k} c_\nu \left\langle C_\nu^{(\alpha)} \left(\vec{\lambda} \right) \right\rangle$$

- The coefficients c_ν are the inverses of Jack characters.

Symmetric Functions

- Symmetric functions are invariant under a permutation of their variables, they are usually indexed by partitions of integers.
- The partitions of 3 ($\nu \perp 3$) are (3), $(2,1)$ and $(1,1,1)$.
- Young diagrams: e.g. (2,1) is represented by 
- Power sum functions:

$$p_{(2,1)}(\lambda_1, \lambda_2, \lambda_3) = \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) \cdot (\lambda_1 + \lambda_2 + \lambda_3)$$

- Monomials:

$$m_{(2,1)}(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_2^2\lambda_3 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2$$

Jack Polynomials

- Jack polynomials:

$$C_\nu^{(\alpha)}\left(\vec{\lambda}\right) \propto m_\nu\left(\vec{\lambda}\right) + \sum_{\sigma \preceq \nu} u_{\nu,\sigma}^{\alpha} m_\sigma\left(\vec{\lambda}\right)$$

- $P(\vec{\lambda})$ appears as the ground-state wave-function squared of a Calogero-Sutherland quantum particle system where β is the inverse temperature.
- The higher energy eigenfunctions are given by:

$$P(\vec{\lambda}) \cdot [\text{linear combinations of Jack polynomials}] \quad \text{with } \alpha = \frac{2}{\beta}$$

- Jack polynomials are orthogonal w.r.t. $P(\vec{\lambda})$ - the average of a Jack polynomial w.r.t. $P(\vec{\lambda})$ is known for the Laguerre, Jacobi and Gaussian ensembles [Baker/Forrester 1997].

Jack Characters

- Jack polynomials and power sum functions are related by a linear transformation:

$$C_{\nu}^{(\alpha)}\left(\vec{\lambda}\right) = \frac{\alpha^{|\nu|}\,|\nu|!}{j_{\nu}} \sum_{\sigma \perp |\nu|} \theta_{\sigma}^{\nu}(\alpha)\, p_{\sigma} \tag{1}$$

- The coefficients $\theta_{\sigma}^{\nu}(\alpha)$ are called Jack characters.
- Jack polynomials with $\alpha = 1$ (i.e. $\beta = 2$) are the Schur functions and $\theta_{\sigma}^{\nu}(1)$ are the characters of the irreducible representations of the symmetric group.

Overview of Derivation

- Begin with the generating function for the power sum polynomials:

$$Q(t) = \sum_{k\geq 1} p_k\left(\vec{\lambda}\right) t^{k-1} = \sum_{i=1}^{N}\sum_{k\geq 1}\lambda_i^k t^{k-1} = \sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{1}{1-\lambda_i t}\right)$$

- Integrate:

$$\sum_{k\geq 1}\frac{p_k\left(\vec{\lambda}\right)}{k} t^k = -\sum_{i=1}^{N}\log\left(\frac{1}{1-\lambda_i t}\right) = -\lim_{u\to 0}\frac{\mathrm{d}}{\mathrm{d}u}\prod_{i=1}^{N}(1-\lambda_i t)^u$$

Overview of Derivation (cont.)

- Now consider the identity:

$$\prod_{i=1}^{N} (1-\lambda_i t)^{-b} = {}_1\mathcal{F}_0^{(\alpha)}\left(b; t^N; \vec{\lambda}\right) = \sum_{\nu} \frac{[b]_\nu^{(\alpha)}}{|\nu|!} t^{|\nu|} C_\nu^{(\alpha)}\left(\vec{\lambda}\right)$$

- Our result follows from a substitution of the above identity.

$$\sum_{k\geq 1} \frac{p_k\left(\vec{\lambda}\right)}{k} t^k = -\lim_{u\to 0} \frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{k\geq 1}\sum_{\nu\perp k} \frac{[-u]_\nu^{(\alpha)}}{|\nu|!} t^{|\nu|} C_\nu^{(\alpha)}\left(\vec{\lambda}\right)\right)$$

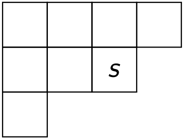
$$p_k\left(\vec{\lambda}\right) = \sum_{\nu\perp k} c_\nu C_\nu^{(\alpha)}\left(\vec{\lambda}\right)$$

The Coefficients c_κ

- The coefficients are:

$$c_\nu = \frac{-1}{(|\nu|-1)!}\left(\lim_{u\to 0}\frac{\mathrm{d}}{\mathrm{d}u}\,[-u]_\nu^{(\alpha)}\right)$$
$$= \frac{1}{(|\nu|-1)!}\prod_{s\in\nu\setminus\{1,1\}}\left(a'(s)-\frac{l'(s)}{\alpha}\right)$$

- For a box s in the Young diagram of ν, $a'(s)$ is the co-arm length and $l'(s)$ is the co-leg length e.g.



; $a'(s) = 2$ and $l'(s) = 1$.

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