Complex Freezing for log-correlated fields

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Plan of the talk

Summary of Gaussian multiplicative chaos theory

- Subcritical Gaussian multiplicative chaos
- Critical Gaussian multiplicative chaos

Freezing in the real phase

Complex case

- Phase I and it's frontier I/II
- Study of phase III and its I/III and II/III boundaries
- Phase II: Freezing conjectures
- Triple point

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• Construct a random measure of the form

$$A \mapsto \int_A e^{\gamma X(x)} dx$$

on a domain D of $\mathbb{R}^d$ where
- X is a log-correlated Gaussian field on D.
- γ is a positive real parameter.

• Different regimes occur depending on the value of γ

• We will mainly focus on two types of log-correlated fields (star scale invariant fields or GFF in $2d$)

Star scale invariant Gaussian fields and $2d$-GFF

Both are Gaussian random distributions with covariance kernel of the form

$$K(x,y) = \int_0^\infty k(u,x,y)\,du$$

- **Star scale invariant kernels:**

$$k(u,x,y) = \frac{k\big(u(x-y)\big)}{u}\mathbf{1}_{u\geq 1}$$

where k is any positive definite continuous function with $k(0) = 1$.

- $2d$**-GFF with Dirichlet b.c.**

$$k(u,x,y) = 2\pi p(u,x,y)$$

where $p(u,x,y)$ is the heat kernel of the Laplacian with Dirichlet b.c.

Remark: in both cases, $\mathbb{E}[X(x)X(y)] \sim \ln \frac{1}{|x-y|}$ for short distances

Regularization

On the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a family of centered Gaussian processes $(X_\varepsilon(x))_{x \in \mathbb{R}^d}$ $(\varepsilon \leq 1)$:

- Covariance: $E[X_\varepsilon(x) X_\varepsilon(y)] \sim \ln \frac{1}{|x-y|+\varepsilon}$
- Filtration $\mathcal{F}_\varepsilon^X = \{X_l(x);\ x \in \mathbb{R}^d,\ \varepsilon \leq l\}$
- for all $\varepsilon < \varepsilon'$, $(X_\varepsilon(x) - X_{\varepsilon'}(x))_{x \in \mathbb{R}^d}$ independent from $\mathcal{F}_{\varepsilon'}$

Remark: GMC is much more general than the framework considered here (see Kahane '85, Robert-Vargas '08, Duplantier-Sheffield '10, Shamov '14, Berestycki '15, Junnila-Saksman '15): see also our review with Vargas

Gaussian multiplicative chaos

Set

$$M_\varepsilon^\gamma(dx) := e^{\gamma X_\epsilon(x)}\,dx$$

Theorem (Kahane, 1985)

There exists a random measure M^γ such that following limit exists almost surely in the space of Radon measures:

$$\varepsilon^{\frac{\gamma^2}{2}} M_\varepsilon^\gamma(dx) \underset{\varepsilon\to 0}{\to} M^\gamma(dx).$$

M^γ is called Gaussian multiplicative chaos associated to the kernel $K(x,y) = \int_0^\infty K(u,x,y)du$.

Gaussian multiplicative chaos

Theorem (Kahane '85)

The measure M^γ is different from 0 if and only if $\gamma^2 < 2d$.

Theorem (Kahane '85)

For $\gamma^2 < 2d$, the measure M^γ "lives" almost surely on a set of Hausdorff dimension $d - \frac{\gamma^2}{2}$ (the γ-thick points).

Critical GMC

Theorem (Duplantier, R., Sheffield, Vargas '12)

There exists a random measure M such that following limit exists almost surely in the space of Radon measures:

$$\varepsilon^d(\sqrt{2d}\ln\frac{1}{\varepsilon} - X_\varepsilon(x))M_\varepsilon^{\sqrt{2d},0}(dx) \underset{\varepsilon\to 0}{\to} M^{'}(dx).$$

$M^{'}$ is called critical Gaussian multiplicative chaos associated to the kernel K.

Critical GMC

Theorem (Duplantier, R., Sheffield, Vargas ’12)

The following limit exists almost surely (along suitable subsequences) in the space of Radon measures:

$$\sqrt{\ln \frac{1}{\varepsilon}}\,\varepsilon^{d} M_{\varepsilon}^{\sqrt{2d}}(dx) \underset{\varepsilon \to 0}{\to} \sqrt{\frac{2}{\pi}} M^{'}(dx)$$

where M' is defined as the almost sure limit in the space of Radon measures

$$M^{'}(dx) := \lim_{\epsilon \to 0} \varepsilon^{d} (\sqrt{2d} \ln \frac{1}{\varepsilon} - X_{\varepsilon}(x)) M_{\varepsilon}^{\sqrt{2d},0}(dx).$$

$M^{'}$ is called critical Gaussian multiplicative chaos associated to the kernel K.

Open question:

- general construction for M' (convolution,...)
- general statement about the above equivalence

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Freezing for star scale invariant kernels

Theorem (Madaule, R., Vargas '13)

For any $\gamma > \sqrt{2d}$, we have the cv in law in the space of Radon measures

$$(\ln \frac{1}{\epsilon})^{\frac{3\gamma}{2\sqrt{2d}}} \epsilon^{\gamma\sqrt{2d}-d} M_\gamma^\epsilon(dx) \to S_\gamma$$

where S_γ is a purely atomic random measure such that, conditionally on M', S_γ is an independently scattered random measure such that for all Borelian $A \subset \mathbb{R}^d$

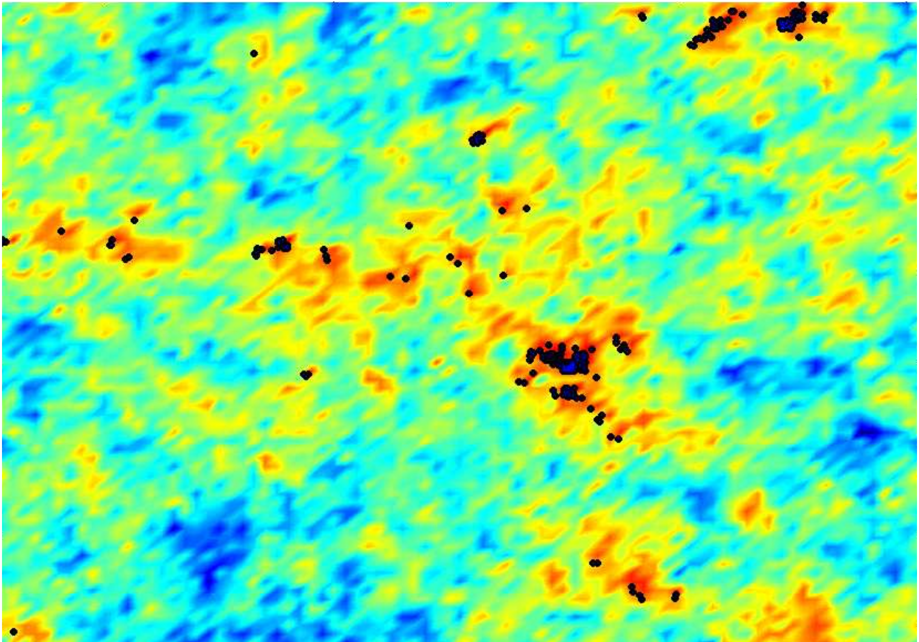
$$\mathbb{E}\left(\exp(-\theta S_\gamma(A))\right) = \mathbb{E}\left(\exp(-\theta^{\frac{\sqrt{2d}}{\gamma}} C(\gamma) M'(A))\right)$$

Remarks:

- S_γ is thus a "$\frac{\sqrt{2d}}{\gamma}$-stable random measure with spatial intensity M' ".

-Fixing an open bounded set D, the size-reordered atoms of the measure $\frac{S_\gamma(dx)}{S_\gamma(D)}$ form a Poisson Dirichlet variable.
(Conjectured by Carpentier-Le Doussal '01, proved by Arguin-Zindy '12)

Freezing for star scale invariant kernels



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Gaussian multiplicative Chaos

We want to give a precise meaning to distributions $M^{\gamma,\beta}$ defined formally by:

$$M^{\gamma,\beta}(A) = \int_A e^{\gamma X(x) + i\beta Y(x)} dx, \quad A \subset \mathbb{R}^d$$

where X, Y two centered **independent** "Gaussian fields" with covariance given by:

$$E[X(x)X(y)] = \int_0^\infty K(u,x,y) du$$

with γ, β nonnegative parameters.

For this, we study the limit of the martingale

$$M_\epsilon^{\gamma,\beta}(dx) := e^{\gamma X_\epsilon(x) + i\beta Y_\epsilon(x)} dx$$

Complex Gaussian multiplicative chaos: Phase diagram

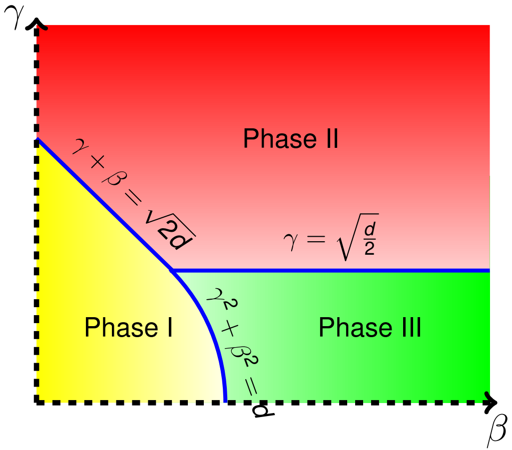

Figure: Phase diagram

Previous works on the topic

Previous work on the complex case:

- Computation of the Free Energy of complex multiplicative cascades: Derrida, Evans, Speer, 1993. In our context

$$\lim_{\varepsilon \to 0} \frac{\ln |\int_{[0,1]^d} M_\varepsilon^{\gamma,\beta}([0,1]^d)|}{\ln \frac{1}{\varepsilon}}$$

- Complex multiplicative cascades: series of works by Barral, Jin, Mandelbrot, 2010. Essentially investigated phase I. Partial results in phase III.

Phase I and it's frontier I/II

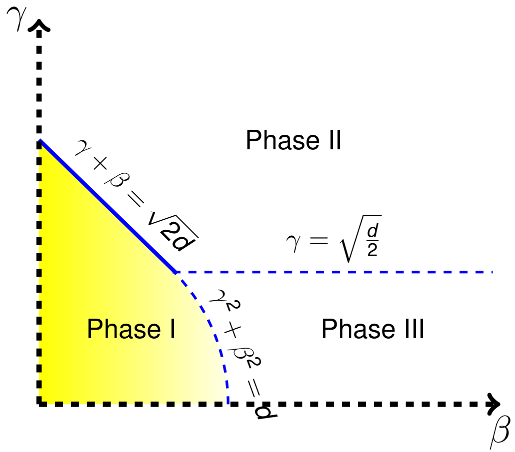

Figure: Phase diagram

Convergence in phase I and it's frontier I/II

Theorem (Lacoin, R., Vargas '13)

Almost sure convergence of the martingale

$$(\epsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} M_\epsilon^{\gamma,\beta}(dx))_\epsilon$$

in the space of distributions of order d towards a non trivial limit $M^{\gamma,\beta}$.

The proof relies on $\mathbb{L}_p$ bounds for energy integrals of GMC.

Open question: Can one show the same statement for correlated X, Y?

Phase III and it's frontier I/III

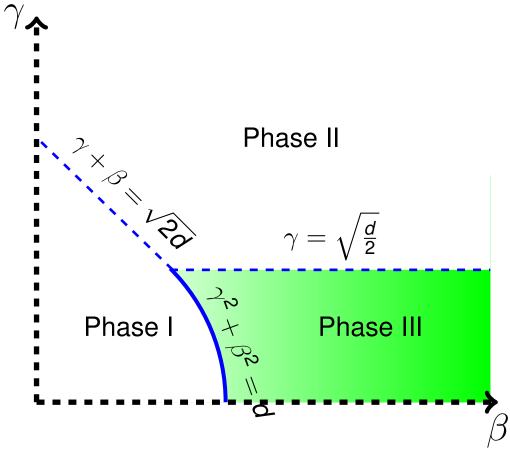

Figure: Phase diagram

Convergence in the inner phase III and it's frontier I/III

Theorem (Lacoin, R., Vargas '13)

- *When* $\gamma \in [0, \sqrt{\frac{d}{2}}[$ *and* $\beta^2 + \gamma^2 > d$*, we have*

$$\left(\epsilon^{\gamma^2 - \frac{d}{2}} M_\epsilon^{\gamma,\beta}(A)\right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 M^{2\gamma}}(A)\right)_{A \subset \mathbb{R}^d}. \quad (1)$$

where $\sigma^2 := \sigma^2(\beta^2 + \gamma^2)$ *and* W *is a standard complex Gaussian measure on* $\mathbb{R}^d$ *with intensity* $\sigma^2 M^{2\gamma}$*.*

- *When* $\gamma \in [0, \sqrt{\frac{d}{2}}[$ *and* $\beta^2 + \gamma^2 = d$*, we have*

$$\left(\epsilon^{\gamma^2 - \frac{d}{2}} |\log \epsilon|^{-1/2} M_\epsilon^{\gamma,\beta}(A)\right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 M^{2\gamma}}(A)\right)_{A \subset \mathbb{R}^d}. \quad (2)$$

where $\sigma^2 = \sigma^2(d)$ *and* W *is a standard complex Gaussian measure on* $\mathbb{R}^d$ *with intensity* $\sigma^2 M^{2\gamma}$*.*

Frontier phase II/III

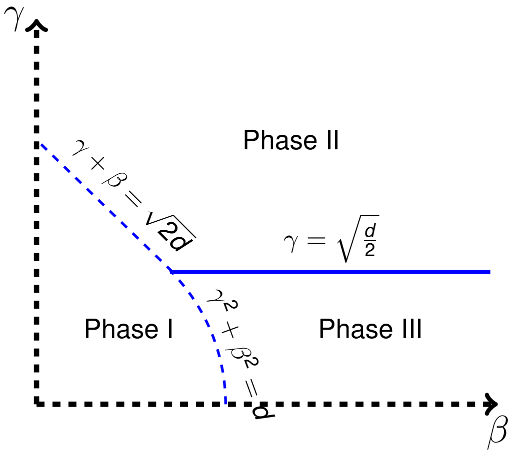

Figure: Phase diagram

Convergence in the frontier phase II/III

Theorem (Lacoin, R., Vargas '13)

When $\gamma = \sqrt{d/2}$ and $\beta^2 + \gamma^2 > d$, we have

$$\left(|\ln \epsilon|^{1/4} M_\epsilon^{\gamma,\beta}(A)\right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 M'}(A)\right)_{A \subset \mathbb{R}^d}.$$

with $\sigma^2 := \sigma^2(\beta)$ and the law of $W_{\sigma^2 M'}(\cdot)$ is, conditionally to X, that of a complex Gaussian random measure with intensity $\sigma^2 M'$.

Phase II

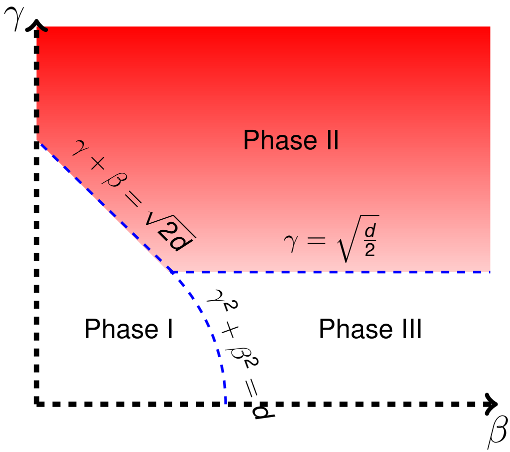

Figure: Phase diagram

Freezing Conjecture on inner phase II: $\beta > 0$

Conjecture

Let $\beta > 0$ and $\gamma > \sqrt{\frac{d}{2}}$ such that $\gamma + \beta > \sqrt{2d}$. Then we get the following convergence in law:

$$\left((\ln \frac{1}{\epsilon})^{\frac{3\gamma}{2\sqrt{2d}}} \epsilon^{\gamma\sqrt{2d}-d} M_\epsilon^{\gamma,\beta}(A) \right)_{A\subset\mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 N^\alpha_{M'}}(A) \right)_{A\subset\mathbb{R}^d}.$$

where $W_{\sigma^2 N^\alpha_{M'}}$ is a complex Gaussian random measure with intensity $N^\alpha_{M'}$ and $N^\alpha_{M'}$ is a α-stable random measure with intensity M' and $\alpha = \sqrt{\frac{d}{2}}\frac{1}{\gamma}$.

The constant σ^2 depends on (γ, β).

Triple point

Conjecture

For $\beta = \gamma = \sqrt{d/2}$ the following convergence in law should hold:

$$\left((\ln \frac{1}{\epsilon})^{-\frac{1}{4}} M_{\epsilon}^{\gamma,\beta}(A) \right)_{A \subset \mathbb{R}^d} \Rightarrow (W_{\sigma^2 M'}(A))_{A \subset \mathbb{R}^d} .$$

where $W_{\sigma^2 M'}$ is a complex Gaussian random measure with intensity M'.

Further question: What are the continuity properties around the triple point?

Thanks!

Capacity analysis for phase diagram

Capacity $\iint |x - x'|^{-\beta^2} M_\gamma(dx) M_\gamma(dx')$

Multifractal analysis on diagonal: Let T_α be the set of α-thick points, i.e. $X_\epsilon(x) \sim \alpha \ln \frac{1}{\epsilon}$. We have $\# T_\alpha \sim (1/\epsilon)^{d - \frac{\alpha^2}{2}}$

On-diagonal contribution of α-thick points

$$\sum_{x \in T_\alpha} \epsilon^{-\beta^2} (M_\gamma(B(x,\epsilon)))^2 = \epsilon^{f(\alpha)}$$

with

$$f(\alpha) = d + \frac{\alpha^2}{2} - 2\gamma\alpha + \gamma^2 - \beta^2$$

- if $\gamma < \sqrt{d/2}$ min of f achieved at $\alpha = 2\gamma$ in which case main contribution is $\epsilon^{d - \beta^2 - \gamma^2} \Rightarrow$ transition at $\beta^2 + \gamma^2 = d$
- if $\gamma > \sqrt{d/2}$ min of f achieved at $\alpha = \sqrt{2d}$ in which case main contribution is $\epsilon^{(\gamma - \sqrt{2d})^2 - \beta^2} \Rightarrow$ transition at $\beta + \gamma = \sqrt{2d}$