# Multiplicative chaos measures for a random model of the Riemann zeta function

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Based on joint work with **C. Webb**

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where $\omega$ is random, distributes uniformly on $[0,1]$, and $T >> 1$. In the limit $T \to \infty$ the quantities

$$p^{-i\omega T}, \qquad p = 2,3,5,7,\ldots$$

become independent uniform distributions on $\mathbb{T} = \{|z| = 1\}$.

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where $p_j$:s are primes listed in an increasing order. We get a well-behaved approximation (our basic object) by considering the partial sum

$$X_N(x) = \sum_{j=1}^{N} \frac{1}{\sqrt{p_j}} \left(\cos(x \log p_j) \cos\theta_{p_j} + \sin(x \log p_j) \sin\theta_{p_j}\right).$$

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- By martingale property, the existence of the limit is easy! Non-triviality and properties of the limit are not so obvious.

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• *Also $\mu_c$ is absolutely continuous with respect to a critical Gaussian multiplicative chaos measure.*

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- $E_N$ *is smooth and almost surely* $E_N \to E$ *uniformly, where* $E$ *is almost surely continuous on* $(0,1)$.
- *The maximal error in the approximation has finite exponential moments:*

$$\mathbb{E}\exp\left(\lambda \sup_{N\geq 1, x\in[0,1]} |E_N(x)|\right) < \infty \quad \text{for all} \quad \lambda > 0.$$

# Some ideas of the proofs

Proof of Thm 3:

- Divide $X_n$ into blocks $X_n(x) \sim \sum_{m=1}^{m(n)} Y_m(x)$, where

$$Y_m(x) = \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} \left(\cos(x \log p_k) \cos \theta_{p_k} + \sin(x \log p_k) \sin \theta_{p_k}\right).$$

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Approximate the $m$:th block by 'freezing' the factors $\log(p_k)$:

$$Y_m(x) \approx \widetilde{Y}_m(x) := \cos(x \log p_{r_m}) \left[ \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} \cos\theta_{p_k} \right] + \sin(x \log p_{r_m}) \left[ \sum_{k=r_m}^{r_{m+1}-1} \frac{1}{\sqrt{p_k}} \sin\theta_{p_k} \right]$$

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- The sup-norms of the errors induced are approximated by suitable Sobolev norms. The final choice for the subsequence $(r_m)$ is $r_m := e^{m^{1/3}}$.

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- We verify that a 'scaling invariant' 1-dim log-correlated field works for the above.
- Finally, to prove the convergence of the Gaussian part one applies a uniqueness result for critical chaos in [Junnila-S 2015], and the rest is not difficult.

Thanks for your patience!