The correction term for the maximum of the log characteristic polynomial of CUE

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The CUE field

1. For $|z| \leq 1$,

$$\mathbf{U}(z) := \mathbf{U}_N(z) = \sum_{h=1}^{N} \log|1 - ze^{i\theta_h}|,$$

where $\{e^{i\theta_h}\}_{h=1}^{N}$ are the eigenvalues of an $N \times N$ Haar unitary matrix.

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2. Maximum principle:

$$\sup_{|z|<1} \mathbf{U}(z) = \sup_{|z|=1} \mathbf{U}(z),$$

CUE Theorems

Theorem (Arguin–Belius–Bourgade: 1511.07399)

The law of large numbers:

$$\frac{\max_{|z|=1} \mathbf{U}_N(z)}{\log N} \xrightarrow[N\to\infty]{\mathbb{P}} 1,$$

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Theorem (P'–Zeitouni: 1602.08875)

The correction term:

$$\frac{\max_{|z|=1} \mathbf{U}_N(z) - \log N}{\log \log N} \xrightarrow[N\to\infty]{\mathbb{P}} -\frac{3}{4},$$

The Gaussian field

Let $\mathcal{W}$ be

$$\mathcal{W}(\omega) = \text{w-lim}_{N\to\infty} \sum_{h=-N}^{N} \sqrt{h} \cdot Z_h \omega^h,$$

i.i.d. complex normals $\{Z_h\}_{h=1}^{\infty}$, $\mathbb{E}Z_h^2 = 0$, $\mathbb{E}|Z_h|^2 = 1$, $Z_{-h} = \overline{Z}_h$.

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$$\mathbf{G}(z) = \frac{1}{2\pi} \int_0^{2\pi} \log|1 - ze^{i\theta}| \cdot \mathcal{W}(e^{i\theta})\, d\theta.$$

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Then $\mathbf{G}$ is a random Gaussian harmonic function on $\{z : |z| < 1\}$ with $\mathbf{G}(0) = 0$ almost surely and covariance

$$\mathbb{E}\mathbf{G}(z)\mathbf{G}(y) = -\frac{\log|1 - z\bar{y}|}{2}.$$

U and **G**

Fix $\delta > 0$. Fact: considered as a random continuous function on $|z| < (1-\delta)$,

$$\mathbf{U}_N(z) \underset{N\to\infty}{\Longrightarrow} \mathbf{G}(z)$$

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$$\max_{|z|=1-N^{-1}} \mathbf{G}(z) - \log N + \frac{3}{4}\log\log N \underset{N\to\infty}{\Longrightarrow} \xi,$$

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Hope: if $\mathbf{U}_N$ and $\mathbf{G}$ were *close enough* for $|z| \le 1 - N^{-1}$, then their maximums would have the same order.

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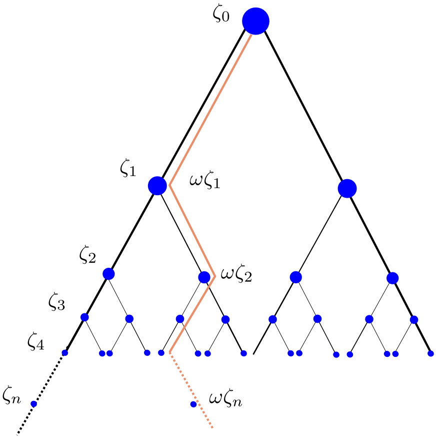
Branching random walk



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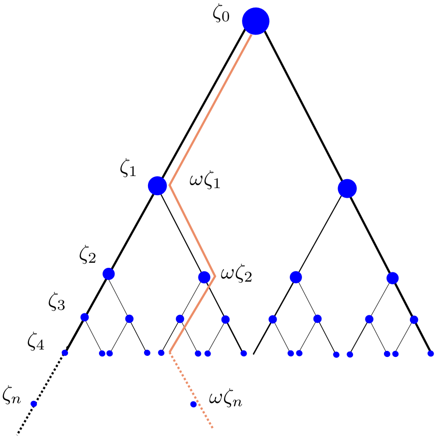

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$$X(\omega\zeta_n) = \sum_{k=1}^{n} Z_{(\omega\zeta_{k-1}, \omega\zeta_k)}.$$

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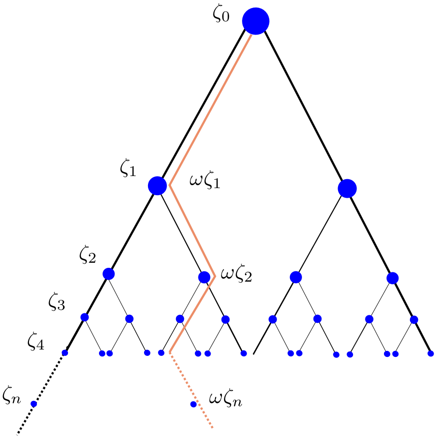

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$$X(\omega\zeta_n) = \sum_{k=1}^{n} Z_{(\omega\zeta_{k-1},\omega\zeta_k)}.$$

3. Every pair $\{(X(\omega_1\zeta_k))_1^\infty, (X(\omega_2\zeta_k))_1^\infty\}$ are Gaussian random walks, equal for the first $(\omega_1, \omega_2)_{\zeta_0}$ steps and independent after.

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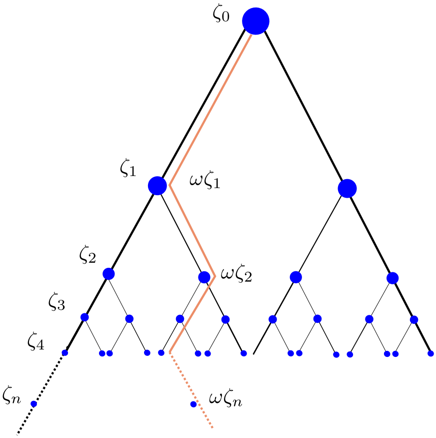

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4. The *barrier method* uses this pair structure to show tightness for

$$\max_\omega X(\omega\zeta_n) - m_n$$

with

$$m_n = c_1 n + c_2 \log n.$$

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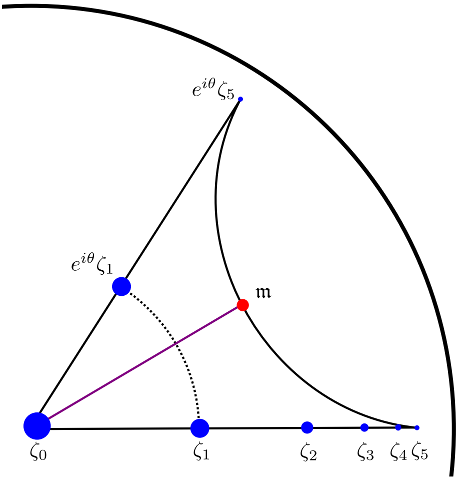
4.

$$\begin{aligned}\mathrm{Var}\left(\mathbf{G}(z) - \mathbf{G}(y)\right) &= \log\left(\cosh\left(\frac{d_{\mathbb{H}}(z,y)}{2}\right)\right) \\ &= \frac{d_{\mathbb{H}}(z,y)}{2} - \log 2 + O(e^{-d_{\mathbb{H}}(z,y)}),\end{aligned}$$

Hyperbolic branching



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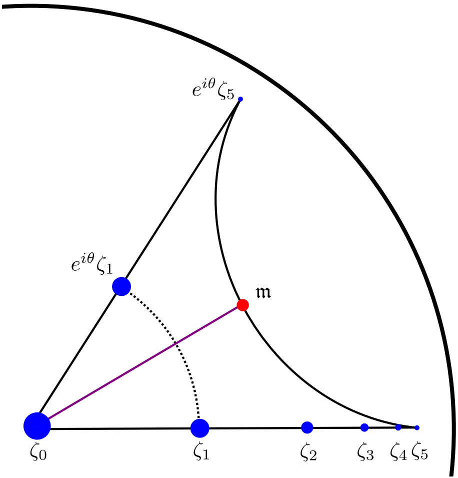

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$$d_{\mathbb{H}}(\zeta_i, \zeta_j) = |i - j|.$$

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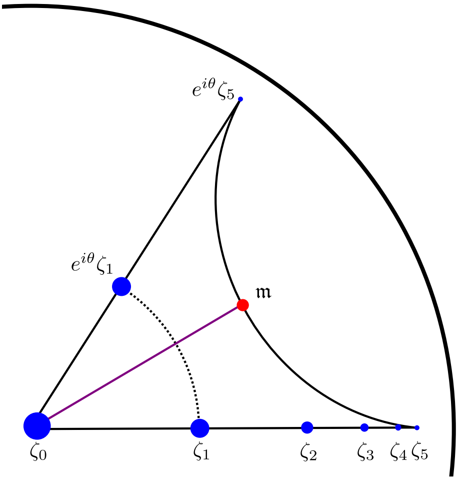

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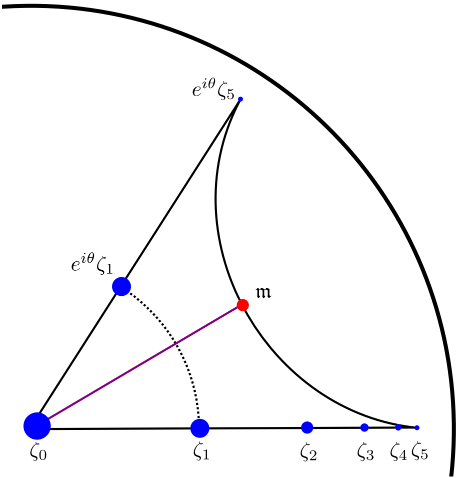

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3. The pair

$$\{(\mathbf{G}(\zeta_k))_1^\infty, (\mathbf{G}(e^{i\theta}\zeta_k))_1^\infty\}$$

are Gaussian random walks, *roughly* coupled for the first $d_{\mathbb{H}}(0, \mathfrak{m}) \approx (1, e^{i\theta})_{\zeta_0}$ steps and roughly independent after.

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Bounding the maximum of $\mathbf{G}$

1. Let $n = [\log N]$, so that *ideally* $\mathbf{U}_N \approx \mathbf{G}$ for z with $d_{\mathbb{H}}(0, z) \leq n$.

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3. We will sketch the argument for

$$\frac{\max_{\omega \in \mathbb{T}_n} \mathbf{G}(\omega \zeta_n) - n}{\log n} \xrightarrow{\mathbb{P}} -\frac{3}{4}$$

using the barrier method.

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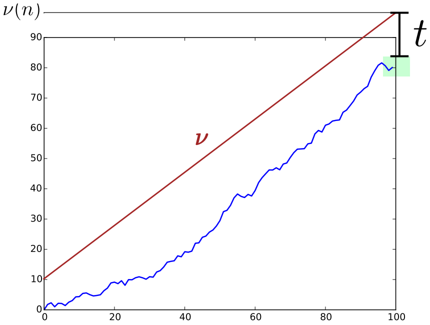

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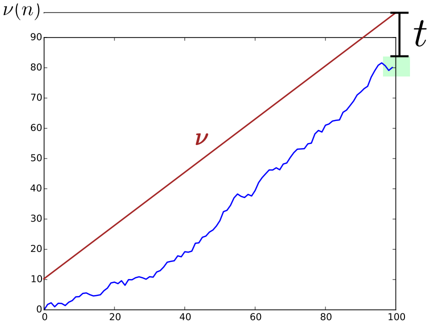

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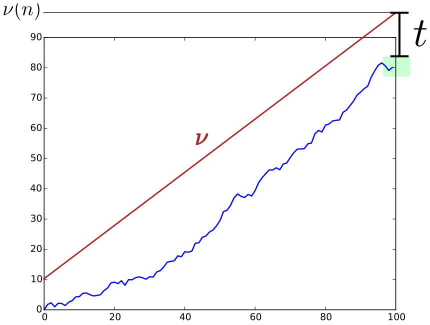

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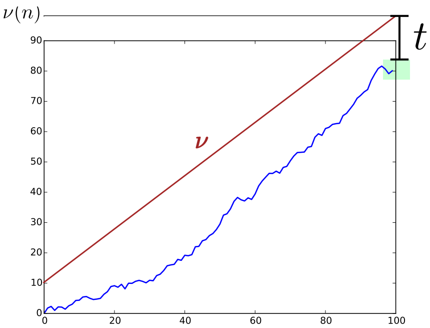

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4. Define $\mathcal{E}(\omega, t)$:

$$\{\mathbf{G}(\omega\zeta_n) - \nu(n) + t \in [0, -1]\}\,.$$

Biasing

Because X is Gaussian,

$$\mathbb{E}\left[f((\mathbf{G}(\zeta_k))_1^n)e^{\alpha\mathbf{G}(\zeta_n)}\right] = \mathbb{E}\left[f((\mathbf{G}(\zeta_k) + \tfrac{\alpha}{2}\mu(k))_1^n)\right]\mathbb{E}[e^{\alpha\mathbf{G}(\zeta_n)}].$$

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$\mu(k) = \mathbb{E}\mathbf{G}(\zeta_k)\mathbf{G}(\zeta_n) \approx k.$

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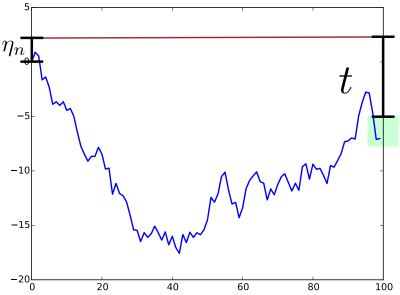

1. With $\alpha = 2 - \frac{3\log n}{2n}$

$$\mathbb{E}\left[\mathbf{1}\left\{\mathcal{B}\cap\mathcal{E}\right\}e^{\alpha\mathbf{G}(\zeta_n)}\right]$$
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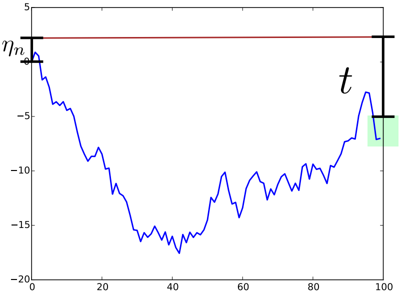

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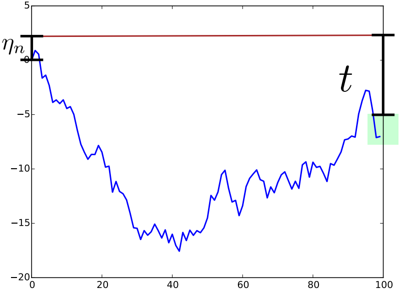

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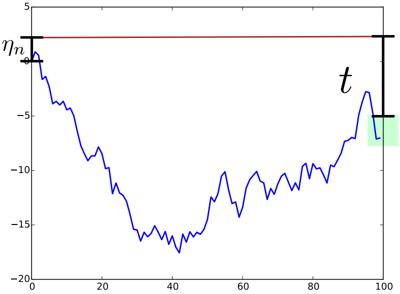

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4. *Ballot theorem*: for $0 \leq t \leq \log n$,

$$\mathbb{P}(\mathcal{B}^F \cap \mathcal{E}^F(t)) = \Theta\left(\frac{\eta_n^2}{n^{3/2}}\right).$$

Upper bound

1. On the one hand,

$$\mathbb{E}\left[\mathbf{1}\left\{\mathcal{B}\cap\mathcal{E}(t)\right\}e^{\alpha\mathbf{G}(\zeta_n)}\right]\asymp\mathbb{E}\left[\mathbf{1}\left\{\mathcal{B}\cap\mathcal{E}(t)\right\}\right]e^{\alpha(n-0.75\log n+\eta_n-t)}$$

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2. On the other hand, using tilting and the ballot theorem,

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3. Putting it together

$$\mathbb{E}\left[\mathbf{1}\{\mathcal{B}(\omega)\cap\mathcal{E}(\omega,t)\}\right] \asymp \eta_n^2 e^{-n-2\eta_n+2t}$$

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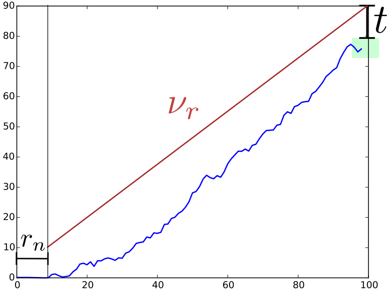
5. With $\eta_n = 0.75\log n$, and $0 < t \leq 0.75\log n - C\log\log n$ and ruling out the complement of the barrier gives

$$\max_{\omega\in\mathbb{T}_n}\mathbf{G}(\omega\zeta_n)\leq n-0.75\log n + C\log\log n$$

with $\mathbb{P}\to 1$.

Lower bound



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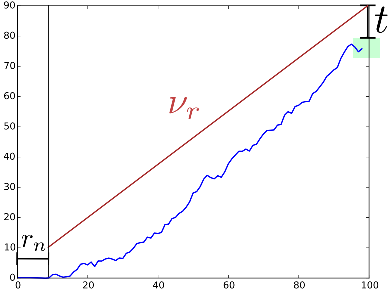

1. Let $r_n \to \infty$ be another sublogarithmic sequence.
2. *Zero out* the first r_n levels:

$$\nu_r(i) = \frac{i - r_n}{n}(n - \tfrac{3}{4}\log n) + \eta_n.$$

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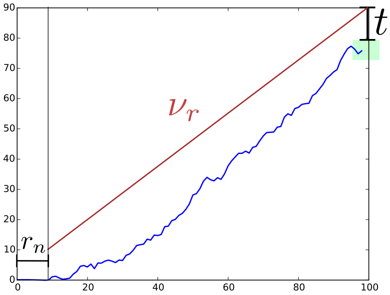

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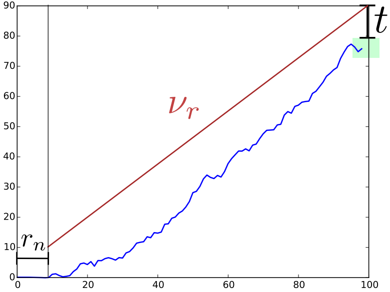

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$$\{\mathbf{G}_r(\omega\zeta_n) - \nu_r(n) + t \in [0, -1]\}.$$

Second moment setup

1. Define

$$\mathfrak{B}_\omega(\mathbf{F}) = 2\mathbf{F}(\omega\zeta_n), \quad \Upsilon(\omega) = e^{\mathfrak{B}_\omega(\mathbf{G}_r)}\mathbf{1}\left\{\mathcal{B}(\omega) \cap \mathcal{E}(\omega)\right\}.$$

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3. We then have

$$\mathbb{E}[\mathbf{1}\{Z > 0\}] \geq \frac{\left(\sum_\theta \mathbb{E}\left[\Upsilon(\omega)\right]\right)^2}{\sum_{\omega_1,\omega_2} \mathbb{E}\left[\Upsilon(\omega_1)\Upsilon(\omega_2)\right]},$$

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4. Conclude:

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Barrier method moral

For finite $\mathbf{z}, \mathbf{w} \subset \mathbb{D}$ and field $\mathbf{F}$, set

$$\mathfrak{B}(\mathbf{F}) = \sum_{z \in \mathbf{z}} 2\mathbf{F}(z) - \sum_{w \in \mathbf{w}} 2\mathbf{F}(w).$$

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It suffices to compute

$$\frac{\mathbb{E}\left[F(\mathbf{G}) e^{\mathfrak{B}(\mathbf{G})}\right]}{\mathbb{E} e^{\mathfrak{B}(\mathbf{G})}}$$

for bounded cylinder functions F with accuracy n^{-100} to conclude

$$\frac{\max_{\omega \in \mathbb{T}_n} \mathbf{G}(\omega \zeta_n) - n}{\log n} \xrightarrow{\mathbb{P}} -\frac{3}{4}$$

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5. Quasi-corollary:

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Basics

Let $f \in \mathrm{L}^1(\mathcal{T})$. Define the *N-th order Toeplitz determinant with symbol* f by

$$D_N(f) = \det\left(\hat{f}(j-k)\right)_{1\le j,k\le N},$$

with $\hat{f}(k)$ the k-th Fourier coefficient of f, i.e.

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Heine identity:

$$\mathbb{E}\left[\prod_{h=1}^{N} f(e^{i\theta_h})\right] = D_N(f).$$

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For some $|a_j| < 1$ and $|b_j| < 1$:

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Also:

$$\prod_{i=1}^{m}\prod_{j=1}^{\ell}\frac{1}{1 - a_j b_i} = \exp\left(\frac{1}{2\pi i}\int_{S^1}\log\left(\frac{1}{V_m(z)}\right)\frac{d}{dz}\log(U_\ell(z^{-1}))\,dz\right).$$

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Generalization

Proposition (P'–Zeitouni)

For any $N \geq \ell > k \geq 0$, any polynomial p nonvanishing on $\{0\} \cup \{a_j\}_{j=1}^{\ell}$ of degree at most N :

$$D_N\left(\frac{p(e^{i\theta})e^{-ik\theta}}{U_\ell(e^{-i\theta})V_m(e^{i\theta})}\right) = (-1)^{\binom{k}{2}+kN}p(0)^{N+k}$$

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Applying Proposition 2,

$$D_N \left(\frac{p(e^{i\theta}) e^{-i\theta}}{U_\ell(e^{-i\theta}) V_m(e^{i\theta})} \right) = \frac{|1 - \bar{z}_1 z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} - \frac{|z_1 - z_2|^2 |z_1|^{2N}}{(1 - |z_1|^2)(1 - |z_2|^2)},$$

for all N sufficiently large.

Main identity

Proposition (P'–Zeitouni)

Let $\mathbf{z}, \mathbf{y} \subset \mathbb{D}$, $|\mathbf{z}| = |\mathbf{y}|$. Let f_1 and f_2 be polynomials of degree strictly less than $N/2$ with $f_1(0) = f_2(0) = 1$ and without zeros in $\bar{\mathbb{D}}$ so that $z \mapsto f_1(z^{-1})$ does not vanish on $\mathbf{z}$.

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$$\cdot \exp\left(\frac{1}{2\pi i}\int_\gamma \log\left(\frac{1}{f_2(z)}\right)\frac{d}{dz}\log(f_1(z^{-1}))\,dz\right)\cdot\frac{\prod_{z\in\mathbf{z}}f_1(z)f_2(\bar{z})}{\prod_{y\in\mathbf{y}}f_1(y)f_2(\bar{y})}$$

$$\cdot\sum_{\substack{S_1,S_2\subset\mathbf{z}\\|S_1|=|S_2|}}(-1)^{|S_1|}\cdot\left[\prod_{S_1}\frac{z^N f_1(z^{-1})}{f_2(z)}\right]\cdot\left[\prod_{S_2}\frac{\bar{z}^N f_2(\bar{z}^{-1})}{f_1(\bar{z})}\right]\cdot c^{\mathbf{y}}(S_1,S_2)\cdot c_{\mathbf{z}}(S_1,S_2).$$

Main identity

Proposition (P'–Zeitouni)

Let $\mathbf{z}, \mathbf{y} \subset \mathbb{D}$, $|\mathbf{z}| = |\mathbf{y}|$. *Let* f_1 *and* f_2 *be polynomials of degree strictly less than* $N/2$ *with* $f_1(0) = f_2(0) = 1$ *and without zeros in* $\bar{\mathbb{D}}$ *so that* $z \mapsto f_1(z^{-1})$ *does not vanish on* $\mathbf{z}$.

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Perturbing the bias

Recall:

Theorem (P'–Zeitouni)

Fix $K > 0$. There exists $m > 100, C > 0$ so that:

1. *for* $\mathbf{z}, \mathbf{w} \subset \mathbb{D}(1 - N^{-1})$;
2. *for* $F \in \mathscr{B}(\{\zeta_i \omega_j \ : \ 1 \le i \le n - m \log n, 1 \le j \le 2\})$;

we have

$$\left| \frac{\mathbb{E}\left[F(\mathbf{U}+\mathbf{Z})e^{\mathfrak{B}(\mathbf{U})}\right]}{\mathbb{E}e^{\mathfrak{B}(\mathbf{U})}} - \frac{\mathbb{E}\left[F(\mathbf{G}+\mathbf{Z})e^{\mathfrak{B}(\mathbf{G})}\right]}{\mathbb{E}e^{\mathfrak{B}(\mathbf{G})}} \right| \le C(\log N)^{-K}.$$

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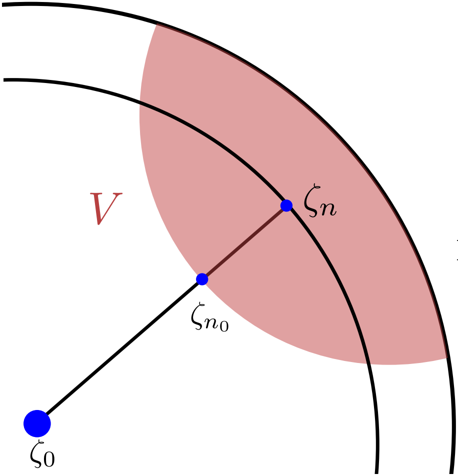
Perturbations of $\mathfrak{B}$ only affect the mesoscopic distribution through a change of mean.

Perturbing the bias



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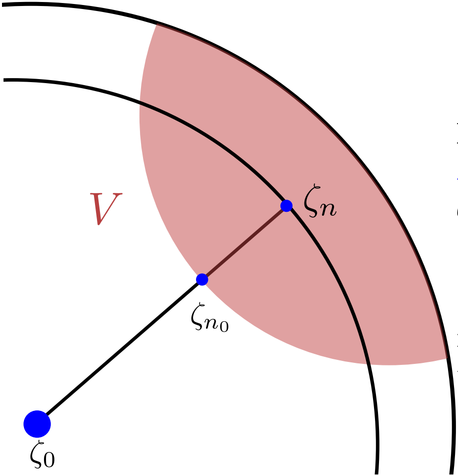

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$$\mathfrak{B}_x(\mathbf{F}) = \mathfrak{B}(\mathbf{F}) + \mathbf{F}(z) - \mathbf{F}(w)$$

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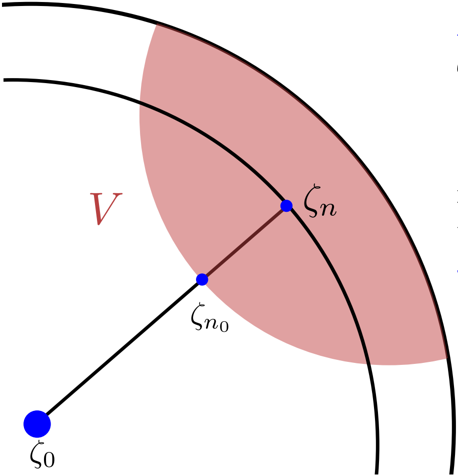

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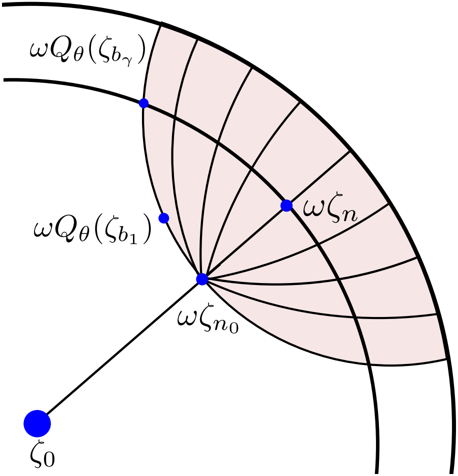
3. Therefore,

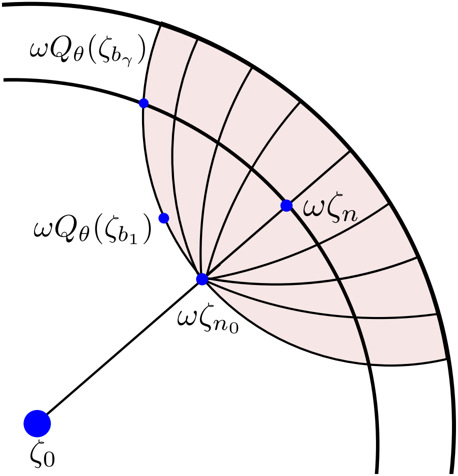
$$\frac{\mathbb{E}\left[F(\mathbf{G}+\mathbf{Z})e^{\mathfrak{B}(\mathbf{G})}\right]}{\mathbb{E}e^{\mathfrak{B}(\mathbf{G})}} = \frac{\mathbb{E}\left[F(\mathbf{G}+\mathbf{Z})e^{\mathfrak{B}_x(\mathbf{G})}\right]}{\mathbb{E}e^{\mathfrak{B}_x(\mathbf{G})}} + o(1)$$

Microscopic landscape



1. Let $\Theta = \{\theta\} \subset [-1, 1]$ be equally spaced of cardinality e^{n-n_0}.

Microscopic landscape



1. Let $\Theta = \{\theta\} \subset [-1, 1]$ be equally spaced of cardinality e^{n-n_0}.
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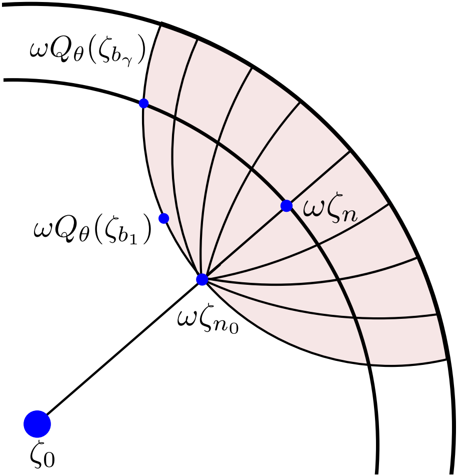

1. Let $\Theta = \{\theta\} \subset [-1,1]$ be equally spaced of cardinality e^{n-n_0}.
2. Let Q_θ be the rotation about $\omega\zeta_{n_0}$ by angle θ.
3. Can we use only exponential moments to show

$$\frac{\max_{\theta\in\Theta} \mathbf{G}(Q_\theta(\zeta_n)) - \mathbf{G}(\zeta_{n_0})}{n - n_0} \xrightarrow{\mathbb{P}} 1?$$

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Final goal

1. By conformal covariance, we are back to bounding the maximum of $\mathbf{G}(\zeta_n)$ *using only exponential moments*.

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3. We will sketch the argument for

$$\frac{\max_{\omega \in \mathbb{T}_n} \mathbf{G}(\omega \zeta_n)}{n} \xrightarrow{\mathbb{P}} 1$$

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Setup

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3. $$\mathcal{E}(\theta) = \{|\mathbf{F}(e^{i\theta}\zeta_n) - b_{\eta-1}| < \eta \cdot n^{1/2}\}$$
$$\mathcal{B}(\theta) = \{|\mathbf{F}(e^{i\theta}\zeta_{b_k}) - b_{k-1}| \leq \eta \cdot n^{1/2},\ \forall\, 1 \leq k \leq \eta - 1\}$$

Second moment setup

Define

$$\mathfrak{B}_\theta(\mathbf{F}) = 2\mathbf{F}(e^{i\theta}\zeta_n) - 2\mathbf{F}(e^{i\theta}\zeta_{b_1}), \Upsilon(\theta) = e^{\mathfrak{B}_\theta(\mathbf{G})}\mathbf{1}\left\{\mathbf{G} \in \mathcal{B}(\theta) \cap \mathcal{E}(\theta)\right\}.$$

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In terms of this indicator, define the biased counting variable

$$Z = \sum_{\substack{h \in \mathbb{Z} \\ |h| \le e^n}} \Upsilon(he^{-n}).$$

Second moment setup

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$$\mathfrak{B}_\theta(\mathbf{F}) = \mathbf{2F}(e^{i\theta}\zeta_n) - \mathbf{2F}(e^{i\theta}\zeta_{b_1}), \Upsilon(\theta) = e^{\mathfrak{B}_\theta(\mathbf{G})}\mathbf{1}\{\mathbf{G} \in \mathcal{B}(\theta) \cap \mathcal{E}(\theta)\}.$$

In terms of this indicator, define the biased counting variable

$$Z = \sum_{\substack{h \in \mathbb{Z} \\ |h| \le e^n}} \Upsilon(he^{-n}).$$

We then have

$$\mathbb{E}[\mathbf{1}\{Z > 0\}] \ge \frac{\left(\sum_\theta \mathbb{E}[\Upsilon(\theta)]\right)^2}{\sum_{\theta_1,\theta_2} \mathbb{E}[\Upsilon(\theta_1)\Upsilon(\theta_2)]},$$

Key estimates

We want that for a $1 - o(1)$ fraction of the pairs (θ_1, θ_2) ,

$$\mathbb{E}\left[Y(\theta_1)Y(\theta_2)\right] \leq \mathbb{E}\left[Y(\theta_1)\right]\mathbb{E}\left[Y(\theta_2)\right](1 + o(1)).$$

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Aside: For (θ_1, θ_2) very close, we also need an estimate that is *not too bad*:

$$\mathbb{E}\left[Y(\theta_1)Y(\theta_2)\right] \lesssim \mathbb{E}\left[Y(\theta_1)\right]\mathbb{E}\left[Y(\theta_2)\right]|\theta_1 - \theta_2|^{-1}$$

This is the reason for the $\mathcal{B}(\theta)$ event.

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Using branching, for $|\theta_1-\theta_2|=\omega(\eta^{-1})$

$$\mathbb{E}\left[e^{\mathfrak{B}_{\theta_1}(\mathbf{G})+\mathfrak{B}_{\theta_2}(\mathbf{G})}\right] = \mathbb{E}\left[e^{\mathfrak{B}_{\theta_1}(\mathbf{G})}\right]\mathbb{E}\left[e^{\mathfrak{B}_{\theta_2}(\mathbf{G})}\right](1+o(1)).$$

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Combining everything, for most pairs (θ_1, θ_2):

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$$\mu_\theta(e^{i\theta}b_j) := \frac{\mathbb{E}\left[e^{\mathfrak{B}_\theta(\mathbf{G})}\mathbf{G}(e^{i\theta}b_j)\right]}{\mathbb{E}\left[e^{\mathfrak{B}_\theta(\mathbf{G})}\right]} \approx b_{j-1}.$$

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Recall, $Y(\theta) > 0$ requires all $|\mathbf{G}(e^{i\theta}\zeta_{b_j}) - b_{j-1}| \leq \eta n^{1/2}$.
Ideally, we could estimate:

$$\mathbb{E}\left[e^{\mathfrak{B}_\theta(\mathbf{G})}\mathbf{G}(e^{i\theta}\zeta_{b_j})^k\right]$$

for $k = 1, 2$ and $j = 1, 2, \ldots, \eta$.

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$$\sum_{j=1}^{k}\mathbb{E}\left[e^{\mathfrak{B}_\theta(\mathbf{G})}(\mathbf{G}(ie^{i\theta}\zeta_{t_j}) - \mathbf{G}(ie^{i\theta}\zeta_{t_{j-1}}))\right].$$

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5. A similar technique can be used for the second moments and lower bounds.

Side comment

Side theorem: using these techniques, as well as the Fyodorov-Strahov identity and DKMVZ orthogonal polynomial asymptotics:

Theorem (Lambert–P' 16+)

Let $\mathbf{L}_N(z)$ be the centered log-potential of a $\beta = 2$ random matrix, with potential V real analytic on $\mathbb{R}$, one-cut and with regular boundary conditions. Then:

$$\frac{\max_{|z|=1} \mathbf{L}_N(z)}{\log N} \xrightarrow[N\to\infty]{\mathbb{P}} 1,$$

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Thanks!