Linear Spectral Statistics for Half-Heavy-Tailed Wigner Matrices

Anna Maltsev Florent Benaych-Georges

University of Bristol

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Central Limit Theorems

- **Usual CLT** If $X_1, ..., X_N$ are i.i.d. random variables with mean μ and variance σ^2, and $S_N = \sum X_j/N$ then $(S_N - \mu)\sqrt{N}$ converges in distribution to $N(0, \sigma^2)$.

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- **Question:** What if $\sigma^2 = \infty$?

  How heavy is the tail? **Crucial parameter** $\alpha =$ the number of moments ($\in \mathbb{R}^+$) i.e. $\mathbb{P}(X > x) \sim x^{-\alpha}$.

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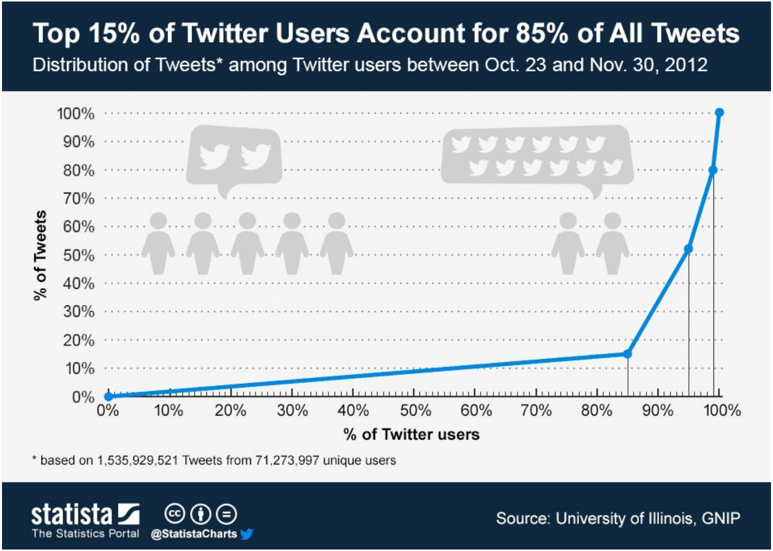
- **Easy version of heavy-tailed CLT** If $X_1, \ldots, X_N$ are i.i.d. heavy-tailed with parameter $\alpha < 2$, then

$$\sum X_j = N\,\mathbb{E}\,X_1 + N^{1/\alpha} Z_\alpha + o(N^{1/\alpha}).$$

  Here Z_α is also heavy-tailed with parameter α.
- **Phase transition**
- Analogous phenomena in RMT

Motivation


Top 15% of Twitter Users Account for 85% of All Tweets
Distribution of Tweets* among Twitter users between Oct. 23 and Nov. 30, 2012
% of Tweets
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
% of Twitter users
* based on 1,535,929,521 Tweets from 71,273,997 unique users
statista
The Statistics Portal
@StatistaCharts
Source: University of Illinois, GNIP

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- Finance: "Half of world's wealth now in hands of 1% of population report" – the Guardian
- Catastrophic phenomena: earthquakes, wars

Definition

Let h be a probability distribution on $\mathbb{C}$. A **Wigner matrix** is a $N \times N$ matrix that is:

- Hermitian,
- entries are iid complex *random variables* with distribution h,
- mean 0 and variance $1/\mathrm{N}$.

The variance is chosen so the spectrum has compact support.

Universality

Consider a Wigner matrix (Hermitian $N \times N$ matrix with iid entries drawn from *any* distribution h). Let

- $\mathcal{N}[a;b]$ = number of eigenvalues in the interval $[a;b]$
- ρ_{sc} is a probability density on $\mathbb{R}$

$$\rho_{sc}(E) = \begin{cases} \frac{1}{2\pi}\sqrt{1-\frac{E^2}{4}}, & \text{if } |E| \leq 2 \\ 0 & \text{if } |E| > 2 \end{cases} .$$

Wigner's key result – universality:
For $\delta > 0$ and all a, b, Wigner proved that

$$\lim_{N\to\infty} P\left(\left|\frac{\mathcal{N}[a;b]}{N(b-a)} - \frac{1}{(b-a)}\int_a^b \rho_{sc}(s)d\,s\right| \geq \delta\right) = 0$$

Transitions

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- Wigner's key result holds when h has 2 moments – Pastur, Girko.
- **Dramatic change** when $\alpha < 2$ (Ben Arous - Guionnet). For $\alpha < 2$, let
$$a_N = \inf\{u : \mathbb{P}(|x_{ij}| > u) \leq 1/N\} \approx N^{\frac{1}{\alpha}}$$
and $A_N = a_N^{-1} X_N$ where X_N is unscaled (i.e. $\mathbb{E}\,|x_{ij}|^2 = 1$).

  Then $\mu_{A_N} := \frac{1}{N} \sum \delta_{\lambda_i} \to \mu_\alpha$.

  Here μ_α is "mostly smooth" and also heavy-tailed with parameter α.

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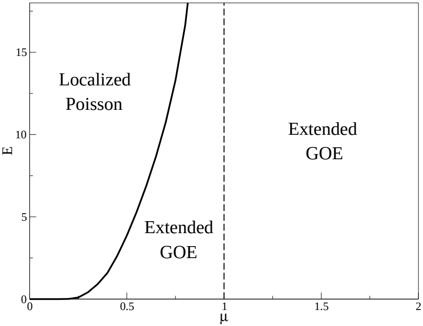
- Using an analysis of stability of localized phase, obtain a closed equation for the mobility edge with numerical solution:



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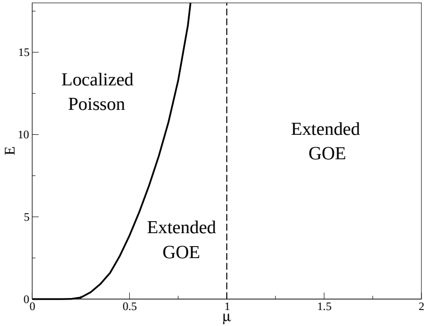

- Supersymmetric approach to show that GOE statistics and Poisson statistics persist throughout the respective phase

Rigorous results in the heavy-tailed case

- For $\alpha < 2$, delocalization at small energies
- For $0 < \alpha < 1$ Bouchaud-Cizeau conjecture a mobility edge
- For $\alpha < 2/3$, localization at large energies
- For $0 < \alpha < 2$, Poisson statistics at the edges

Subtle transition point at $\alpha = 4$

- A lot of proof techniques fail
- Top eigenvalue sticks to the bulk if $\alpha > 4$ and does not otherwise
- Rate of convergence to semicircle law and local behavior of eigenvalues (probably) changes
- No rigorous results in this regime
- Fluctuation of linear spectral statistics changes (and trace of resolvent) changes – rest of my talk

Linear spectral statistic: For a function φ in a class of "nice" test functions, the linear spectral statistic is $\frac{1}{N}\sum_\alpha \varphi(\lambda_\alpha)$ where λ_α are the eigenvalues of the matrix. For $\varphi(x) = \frac{1}{x-z}$ the linear spectral statistic is the normalized trace of the resolvent (Stieltjes transform).

Useful table

Let $G = (A_N - z)^{-1}$. Then $\frac{1}{N}\operatorname{Tr} G = \frac{1}{N}\sum \frac{1}{\lambda_\alpha - z}$.

	$\alpha < 2$	$2 < \alpha < 4$	$\alpha > 4$
Order of $\frac{1}{N}(\operatorname{Tr} G - \mathbb{E}\operatorname{Tr} G)$	$N^{-1/2}$	$N^{-\alpha/4}$	N^{-1}
Description	independent-like	intermediate regime	highly correlated
People	Benaych -Georges Guionnet Male	Benaych -Georges M.	Bai

Our matrices

We say a random variable is **half-heavy-tailed** if for a certain $\alpha \in (2,4)$ and a certain $c > 0$, as $x \to +\infty$,

$$\mathbb{P}(|x_{ij}| > x) \sim \frac{c}{\Gamma(\alpha+1)} x^{-\alpha}. \tag{1}$$

Here $A \sim B$ means that $A/B \longrightarrow 1$ as $N \to \infty$.

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Let

$$A = [a_{ij}]_{1 \le i,j \le N} = \left[\frac{x_{ij}}{\sqrt{N}} \right]_{1 \le i,j \le N},$$

where one of two conditions holds: either

(a) x_{ij}'s, $1 \le i \le j$, are i.i.d. real random variables with $\mathbb{E}\, x_{ij} = 0$, $\mathbb{E}\,|x_{ij}|^2 = 1$, and (1).

or

(b) $x_{ij} = x_{ij}^R/\sqrt{2} + i x_{ij}^I/\sqrt{2}$ for $1 < i < j$ and $x_{ii} = x_{ii}^R$ where x_{ij}^I and x_{ij}^R are i.i.d. real symmetric random variables $\mathbb{E}\, x_{ij} = 0$, $\mathbb{E}\,|x_{ij}|^2 = 1$, and (1)

Our test functions

We prove convergence to a Gaussian process for any random variable of the form

$$\frac{1}{N^{1-\alpha/4}}(\operatorname{Tr}\varphi(A) - \mathbb{E}\operatorname{Tr}\varphi(A)),$$

where φ is a function of the type

$$\varphi(x) = \sum_{j=1}^{p} \frac{c_j}{z_j - x}$$

for some $p \geq 1,\ c_1, \ldots, c_p \in \mathbb{C},\ z_1, \ldots, z_p \in \mathbb{C}\backslash\mathbb{R}$.

Theorem(Benaych-Georges – M.)

The process

$$\left(\frac{1}{N^{1-\alpha/4}}(\operatorname{Tr} G(z) - \mathbb{E}\operatorname{Tr} G(z))\right)_{z\in\mathbb{C}\setminus\mathbb{R}}$$

converges to a complex Gaussian centered process $(X_z)_{z\in\mathbb{C}\setminus\mathbb{R}}$ with covariance defined by the fact that $X_{\overline{z}} = \overline{X_z}$ and that for any $z, z' \in \mathbb{C}\setminus\mathbb{R}$,

$$\mathbb{E}[X_z X_{z'}] =$$

$$-\iint_{t,t'>0} \partial_z \partial_{z'} \Big\{ [(K(z,t) + K(z',t'))^{\frac{\alpha}{2}} - (K(z,t)^{\frac{\alpha}{2}} + K(z',t')^{\frac{\alpha}{2}})]$$

$$\exp\left(\operatorname{sgn}_z itz - K(z,t) + \operatorname{sgn}_{z'} it'z' - K(z',t')\right) \Big\} \frac{c\,\mathrm{d}t\,\mathrm{d}t'}{2tt'}$$

where c and α are as in (1), $\operatorname{sgn}_z := \operatorname{sgn}(\Im z)$ and $K(z,t) := \operatorname{sgn}_z itG_{sc}(z)$, and $G_{sc}(z)$ is the Stieltjes transform of the semicircle law with support $[-2, 2]$.

Proof Outline

1. Truncate, renormalize, and, in the real case, centralize
2. Use the Martingale decomposition and the Central Limit Theorem for Martingales
3. Use resolvent identities
4. Show that the off-diagonal terms of the resolvent do not contribute to the limit
5. Compute the limit with just the diagonal terms

Some Details

To truncate:

- We use $|\operatorname{Tr}(G_B(z) - G_A(z))| \leq 2|\Im z|^{-1} \operatorname{rank}(B - A)$.
- Let $B = [a_{ij}\mathbf{1}_{|x_{ij}| \leq N^\beta} - \mu_N/\sqrt{N}]$. We will solve for β. Subtracting $\mu_N/\sqrt{N}$ from each matrix entry is a rank 1 perturbation.

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- Then, as $\mathbb{P}(|x_{ij}| > N^\beta) \leq CN^{-\alpha\beta}$, we have $\operatorname{rank}(B - A) \leq 1 + 2\sum_{i=1}^{N} X_i$ where the X_i's are independent Bernoulli r.v. with parameters $\mathbb{P}(X_i = 1) = 1 - (1 - CN^{-\alpha\beta})^i$.
- Need $(2 - \alpha\beta)_+ < 1 - \alpha/4$, *i.e.* $\beta > \frac{2-(1-\alpha/4)}{\alpha} = \frac{1}{4}(1 + \frac{\alpha}{4})$.

Martingales for Wigner matrices

- If $\mathcal{F}_k(N) := \sigma(x_{i,j}\,;\, i \leq k \text{ and } j \leq k)$, $\mathbb{E}_k$ is the expectation $\mathbb{E}[\cdot|\mathcal{F}_k]$ and $G(z)$ is the resolvent then

$$Y_k = \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1}) \operatorname{Tr} G$$

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- Use CLT for Martingales
- $\sum Y_k = \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1})\operatorname{Tr} G = \frac{1}{N^{1-\alpha/4}}(\operatorname{Tr} G - \mathbb{E}\operatorname{Tr} G)$ and the variance is $\sum |Y_k|^2$

Key technical ideas

- $Y_k = \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1}) \operatorname{Tr}(G - G^{(k)})$
- Rewrite Y_k's as a log derivative; then by Cauchy Inequality suffices to bound the log

$$\begin{aligned} Y_k &= \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1}) \frac{1 + \mathbf{a}_k^*(G^{(k)}(z))^2 \mathbf{a}_k}{z - a_{kk} - \mathbf{a}_k^* G^{(k)}(z)\mathbf{a}_k} \\ &= \partial_z \log |z - a_{kk} - \mathbf{a}_k^* G^{(k)}(z)\mathbf{a}_k|^2 \end{aligned}$$

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- Crucial difference between diagonal and off-diagonal terms in $\mathbf{a}_k^* G^{(k)} \mathbf{a}_k$: get the power in $a_{kj} G_{ji} \overline{a_{ik}}$ but the second power in $G_{jj}|a_{kj}|^2$

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- Remove off-diagonal terms:

$$\tilde{Y}_k := \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1}) \frac{1 + \mathbf{a}_k^*(G^{(k)}(z))^2_{\text{diag}} \mathbf{a}_k}{z - \mathbf{a}_k^* G^{(k)}(z)_{\text{diag}} \mathbf{a}_k}$$

- Compute the limit with diagonal terms

Computation of the limit

- Need to compute $\mathbb{E}\, f_k := \frac{1+\mathbf{a}_k^*(G^{(k)}(z))^2_{\mathrm{diag}}\mathbf{a}_k}{z-\mathbf{a}_k^* G^{(k)}(z)_{\mathrm{diag}}\mathbf{a}_k}$
- Use $\dfrac{1}{w} = -i\,\mathrm{sgn}_w \displaystyle\int_0^{+\infty} e^{\mathrm{sgn}_w\, itw}\mathrm{d}t$ with $w = z - \sum_j |\mathbf{a}_k(j)|^2 G^{(k)}(z)_{jj}$ to get

$$\mathbb{E}_{\mathbf{a}_k} f_k(z) = -i\,\mathrm{sgn}_z \int_0^{+\infty} \frac{1}{\mathrm{sgn}_z it}\partial_z\, \mathbb{E}_{\mathbf{a}_k}\{e^{\mathrm{sgn}_z it(z-\sum_j |\mathbf{a}_k(j)|^2 G^{(k)}(z)_{jj})}\}\mathrm{d}t$$

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- For $a_k(j)$ are iid so this involves $\mathbb{E}\left[\exp(-i\lambda|a_{11}|^2)\right]$ for $\lambda = \mathrm{sgn}_z tG^{(k)}(z)_{jj}$ which is a Laplace transform – can use to study behavior near 0
- Replace $G^{(k)}(z)$ by G_{sc}

$$\mathbb{E}[X_z X_{z'}] =$$

$$-\iint_{t,t'>0} \partial_z \partial_{z'} \left\{ [(K(z,t) + K(z',t'))^{\frac{\alpha}{2}} - (K(z,t)^{\frac{\alpha}{2}} + K(z',t')^{\frac{\alpha}{2}})] \right.$$
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with $K(z,t) := \mathrm{sgn}_z\, itG_{\mathrm{sc}}(z)$

Remarks

1. Our test functions φ of this type span (by closure) the set of continuous functions $\to 0$ at $\pm\infty$ by the Stone-Weierstrass theorem; would be nice to extend the theorem to analytic or better.
2. Shcherbina's technique does not work for us: We truncate the entries of the matrices to upper-bound the variance of $\mathrm{Tr}(E + i\eta - A)^{-1}$ and this variance does not decay enough as $|E|$ grows.
3. Would be nice to get information on local behavior of eigenvalues. Tricky because large deviation estimates for quadratic forms are poor; also cannot use high moments.

Thank you!