Log-correlated Gaussian fields and Random matrix theory

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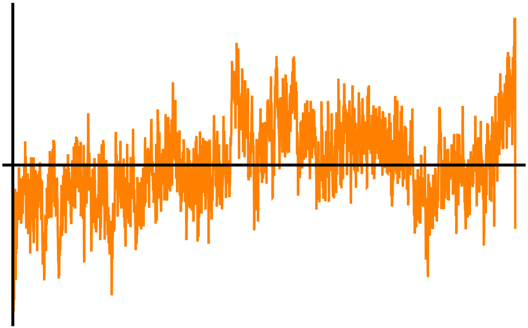
Extrema of logarithmically correlated processes, characteristic polynomials, and the Riemann zeta function, May 9, 2016

What is a log-correlated Gaussian field?

A Gaussian process $X(x)$ on $\mathbb{R}^d$ with a log singularity in its covariance:

$$\mathbb{E}X(x)X(y) \sim -\log|x-y|, \qquad \text{as} \qquad x \to y.$$



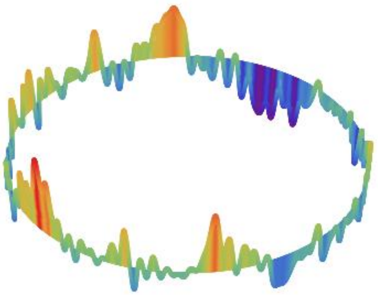
Doesn't make sense as an honest random function.

Examples of log-correlated fields

Let $A_k \sim N_{\mathbb{C}}(0,1)$ be i.i.d. and

$$X(\theta) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left[A_k e^{ik\theta} + A_k^* e^{-ik\theta} \right].$$

Then (formally) $\mathbb{E}X(\theta)X(\theta') = -\frac{1}{2}\log|e^{i\theta} - e^{i\theta'}|$.



Examples of log-correlated fields

- Let $B_k \sim N(0,1)$ be i.i.d. and for $x \in (-1,1)$

$$Y(x) = \sum_{k=1}^{\infty} \sqrt{\frac{1}{k}} B_k T_k(x),$$

where $T_k(\cos\theta) = \cos k\theta$. Then $\mathbb{E}Y(x)Y(y) = -\frac{1}{2}\log(2|x-y|)$.

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- Let $C_k \sim N(0,1)$ be i.i.d. and for $x \in (-1,1)$

$$Z(x) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} C_k U_k(x)\sqrt{1-x^2},$$

where $U_k(\cos\theta) = \sin(k+1)\theta/\sin\theta$. Then

$$\mathbb{E}Z(x)Z(y) = -\log \frac{|x-y|}{1-xy+\sqrt{1-x^2}\sqrt{1-y^2}}.$$

Examples of log-correlated fields

- Let $D_k \sim N(0,1)$ be i.i.d., $D \subset \mathbb{R}^2$ (compact, simply connected) and $\Delta\phi_k = -\lambda_k \phi_k$ on D (zero Dirichlet bc). GFF: for $x \in D$

$$W(x) = \sum_{k=1}^{\infty} \frac{D_k}{\sqrt{\lambda_k}} \phi_k(x),$$

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- All of these series converge almost surely in suitable spaces of generalized functions (e.g. Sobolev spaces) and one can make precise sense of everything above.
- Many other ways of defining log-correlated fields exist.

Log-correlated fields in RMT (and other areas)

- **Moral of the story:** most common CLTs for linear statistics in RMT give rise to log-correlated objects. For nice enough f

$$\sum_{j=1}^{N} f(\lambda_j) - \sum_{j=1}^{N} \mathbb{E} f(\lambda_j) \xrightarrow{d} \mathcal{N}(0, \sigma_f^2).$$

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$$\frac{1}{Z_{N,\beta}} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} e^{-N\frac{\beta}{2}\sum_{j=1}^{N} V(\lambda_j)}$$

Characteristic polynomial of the CUE

- Let $U_N \sim CUE(N)$, and

$$
\begin{aligned}
X_N(\theta) &= \log|\det(I - e^{-i\theta} U_N)| \\
&= -\frac{1}{2}\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}} \left[e^{-ij\theta} \frac{\operatorname{Tr} U_N^j}{\sqrt{j}} + e^{ij\theta} \frac{\operatorname{Tr} U_N^{-j}}{\sqrt{j}} \right].
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- One of the critical ingredients: the CLT of Diaconis and Shahshahani.

Characteristic polynomial of the GUE

- Let $H_N \sim GUE(N)$ (random Hermitian matrix, density $\propto e^{-2N \mathrm{Tr} H_N^2}$) and for $x \in (-1, 1)$.

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- The imaginary part of $\log\det(xI - H_N)$ is essentially the eigenvalue counting function and converges to Z after centering.

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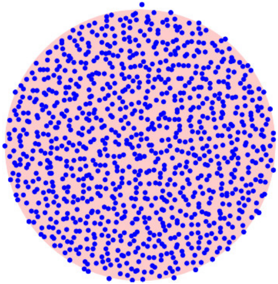
Let G_N be a $N \times N$ complex Ginibre matrix (density $\propto e^{-N\mathrm{Tr}\, G_N^* G_N}$).



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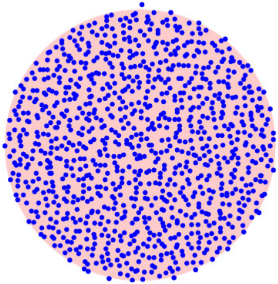

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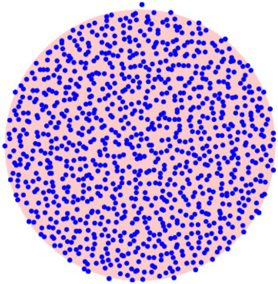

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Theorem (Rider and Virag 06)

As $N \to \infty$, $W_N(z) - \mathbb{E}W_N(z)$ converges to a Gaussian field with covariance $-\frac{1}{2}\log|z-w|$.

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- Similar result for the Plancherel measure due to Kerov (90s).

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- How about Wigner matrices, discrete models, ζ-function, log-gases in $d \geq 3$, ...?
- What's the big picture?
- One guess for a universality class: random analytic functions whose zeroes repel each other like a log-gas on a suitable scale.

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$$\frac{|\det(e^{i\theta}I - U_N)|^\gamma}{\mathbb{E}|\det(e^{i\theta}I - U_N)|^\gamma} \xrightarrow{d} e^{\gamma X(\theta) - \frac{\gamma^2}{2}\mathbb{E}X(\theta)^2}$$

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- Similar results for a discrete CUE (W 15) and GUE (Berestycki, W, and Wong 16).

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$$X_{N,M}(\theta) = -\frac{1}{2}\sum_{j=1}^{M}\frac{1}{\sqrt{j}}\left(e^{-ij\theta}\frac{\mathrm{Tr}\,U_N^j}{\sqrt{j}} + e^{ij\theta}\frac{\mathrm{Tr}\,U_N^{-j}}{\sqrt{j}}\right)$$

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- This lets one show

$$\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{E}\left|\int_0^{2\pi}\left[\frac{e^{\gamma X_N(\theta)}}{\mathbb{E}e^{\gamma X_N(\theta)}} - \frac{e^{\gamma X_{N,M}(\theta)}}{\mathbb{E}e^{\gamma X_{N,M}(\theta)}}\right]d\theta\right|^2 = 0.$$

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