Log-correlated Gaussian fields and Random matrix theory

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Extrema of logarithmically correlated processes, characteristic polynomials, and the Riemann zeta function, May 9, 2016
What is a log-correlated Gaussian field?

A Gaussian process $X(x)$ on $\mathbb{R}^d$ with a log singularity in its covariance:

$$\mathbb{E}X(x)X(y) \sim -\log |x - y|, \quad \text{as} \quad x \to y.$$  

Doesn't make sense as an honest random function.
Examples of log-correlated fields

Let $A_k \sim N_{\mathbb{C}}(0, 1)$ be i.i.d. and

$$X(\theta) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left[ A_k e^{ik\theta} + A_k^* e^{-ik\theta} \right].$$

Then (formally) $\mathbb{E} X(\theta) X(\theta') = -\frac{1}{2} \log |e^{i\theta} - e^{i\theta'}|.$
Examples of log-correlated fields

- Let $B_k \sim N(0, 1)$ be i.i.d. and for $x \in (-1, 1)$

$$Y(x) = \sum_{k=1}^{\infty} \sqrt{\frac{1}{k}} B_k T_k(x),$$

where $T_k(\cos \theta) = \cos k\theta$. Then $\mathbb{E} Y(x) Y(y) = -\frac{1}{2} \log(2|x - y|)$. 
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- Let $C_k \sim \mathcal{N}(0, 1)$ be i.i.d. and for $x \in (-1, 1)$

$$
Z(x) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k + 1}} C_k U_k(x) \sqrt{1 - x^2},
$$

where $U_k(\cos \theta) = \sin(k + 1) \theta / \sin \theta$. Then

$$
\mathbb{E} Z(x) Z(y) = -\log \frac{|x - y|}{1 - xy + \sqrt{1 - x^2} \sqrt{1 - y^2}}.
$$
Examples of log-correlated fields

- Let $D_k \sim N(0, 1)$ be i.i.d., $D \subset \mathbb{R}^2$ (compact, simply connected) and $\Delta \phi_k = -\lambda_k \phi_k$ on $D$ (zero Dirichlet bc). GFF: for $x \in D$

$$W(x) = \sum_{k=1}^{\infty} \frac{D_k}{\sqrt{\lambda_k}} \phi_k(x),$$

Again $\mathbb{E} W(x) W(y) = G_D(x, y) \sim -\log |x - y|$ as $x \to y$. 
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- Many other ways of defining log-correlated fields exist.
Log-correlated fields in RMT (and other areas)

- **Moral of the story:** most common CLTs for linear statistics in RMT give rise to log-correlated objects. For nice enough $f$

$$\sum_{j=1}^{N} f(\lambda_j) - \sum_{j=1}^{N} \mathbb{E} f(\lambda_j) \xrightarrow{d} \mathcal{N}(0, \sigma_f^2).$$
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  - Rider and Virag 06, Ginibre ensemble.
  - Bourgade and Kuan 12, mesoscopic CLT for $\zeta$ zeroes.
  - Borodin, Gorin, and Guionnet 15, discrete $\beta$-ensembles.
  - Moll 15, models for random partitions.
  - 2d $\beta$-ensembles? big picture?
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$$\frac{1}{Z_{N,\beta}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N \frac{\beta}{2} \sum_{j=1}^{N} V(\lambda_j)}$$
Characteristic polynomial of the CUE

Let \( U_N \sim CUE(N) \), and

\[
X_N(\theta) = \log | \det(I - e^{-i\theta} U_N) | \\
= -\frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\sqrt{j}} \left[ e^{-ij\theta} \frac{\text{Tr} U_N^j}{\sqrt{j}} + e^{ij\theta} \frac{\text{Tr} U_N^{-j}}{\sqrt{j}} \right].
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**Theorem (Hughes, Keating, and O'Connell 2001)**

As $N \to \infty$, $X_N \xrightarrow{d} X$ (in a suitable space).
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- Similar result for the imaginary part of $\log \det(I - e^{-i\theta} U_N)$.
- One of the critical ingredients: the CLT of Diaconis and Shahshahani.
Let $H_N \sim \text{GUE}(N)$ (random Hermitian matrix, density $\propto e^{-2N\text{Tr}H_N^2}$) and for $x \in (-1, 1)$.

$$Y_N(x) = \log |\det(xI - H_N)|.$$
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**Theorem (Fyodorov, Khoruzhenko, and Simm 13)**

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- The imaginary part of $\log \det(xI - H_N)$ is essentially the eigenvalue counting function and converges to $Z$ after centering.
Characteristic polynomial of the Ginibre ensemble

Let $G_N$ be a $N \times N$ complex Ginibre matrix (density $\propto e^{-N \text{Tr}G_N^*G_N}$).

Theorem (Rider and Virag 06)

As $N \to \infty$, $W_N(z) - E_{W_N}(z)$ converges to a Gaussian field with covariance $-\frac{1}{2} \log |z - w|$.
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The eigenvalue fluctuation field for the GUE

- Idea: look at $\lambda_j - \mathbb{E}\lambda_j$ "globally".

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$$x = \int_{-1}^{\gamma(x)} \sigma(y) dy.$$
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- It turns out that for $x \in (0, 1)$ (in the sense of generalized functions)

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What’s the big picture?

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- One guess for a universality class: random analytic functions whose zeroes repel each other like a log-gas on a suitable scale.
Gaussian multiplicative chaos

- It turns out one can make sense of things like

\[ e^{\gamma X(\theta) - \frac{\gamma^2}{2} \mathbb{E} X(\theta)^2}. \]
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**Theorem (W 14)**

Let \( U_N \sim \text{CUE}(N) \), and \( 0 < \gamma < \sqrt{2} \). Then

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\frac{|\det(e^{i\theta}I - U_N)|^{\gamma}}{\mathbb{E}|\det(e^{i\theta}I - U_N)|^{\gamma}} \xrightarrow{d} e^{\gamma X(\theta) - \frac{\gamma^2}{2} \mathbb{E}X(\theta)^2}
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Similar results for a discrete CUE (W 15) and GUE (Berestycki, W, and Wong 16).
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Define

\[ X_{N,M}(\theta) = -\frac{1}{2} \sum_{j=1}^{M} \frac{1}{\sqrt{j}} \left( e^{-ij\theta} \frac{\text{Tr} U_j^j}{\sqrt{j}} + e^{ij\theta} \frac{\text{Tr} U_j^{-j}}{\sqrt{j}} \right) \]
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\[ \mathbb{E} e^{\gamma X_N(\theta)}, \mathbb{E} e^{\gamma X_N(\theta) + \gamma X_N(\theta')}, \mathbb{E} e^{\gamma X_N(\theta) + \gamma X_{N,M}(\theta')}, ... \]
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This lets one show

\[ \lim_{M \to \infty} \lim_{N \to \infty} \left| E \int_0^{2\pi} \left[ \frac{e^{\gamma X_N(\theta)}}{E e^{\gamma X_N(\theta)}} - \frac{e^{\gamma X_{N,M}(\theta)}}{E e^{\gamma X_{N,M}(\theta)}} \right] d\theta \right|^2 = 0. \]
Gaussian Multiplicative Chaos

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