# On the maximum of the characteristic polynomial of the Circular Beta Ensemble

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# Presentation of the setting

- We consider the Circular Beta Ensemble (C$\beta$E), corresponding to $n$ points on the unit circle $\mathbb{U}$, whose probability density with respect to the uniform measure on $\mathbb{U}^n$ is given by

$$C_{n,\beta} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^{\beta},$$

for some $\beta > 0$.

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  for some $\beta > 0$.
- For $\beta = 2$, one gets the distribution of the eigenvalues of a Haar-distributed matrix on the unitary group $U(n)$. Other matrix models has been found by Killip and Nenciu in 2004 for general β.

- If $(\lambda_j^{-1})_{1\leq j\leq n}$ are the eigenvalues of a random matrix, one can consider the characteristic polynomial:

  $$X_n(z) = \prod_{j=1}^{n}(1-\lambda_j z),$$

  and its logarithm

  $$\log X_n(z) = \sum_{j=1}^{n}\log(1-\lambda_j z),$$

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  which can be well-defined in a continuous way, except on the half-lines $\lambda_j^{-1}[1, \infty)$.
- We will be interested in the extremal values of $\log X_n(z)$ on the unit circle.

- It can be proven that $\left(\sqrt{\beta/2}\log X_n(z)\right)_{z\in\mathbb{D}}$ ($\mathbb{D}$ being the open unit disc) tends in distribution to a complex Gaussian holomorphic function. This is a consequence of a CLT on the traces of powers of a random matrix corresponding to the C$\beta$E: see results by Diaconis and Shahshahani (1994) for $\beta = 2$, by Jiang and Matsumoto (2011) for general $\beta$.

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- This Gaussian function $\mathbb{G}$ has the following covariance structure:

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$$\mathbb{E}[\overline{\mathbb{G}(z)}\mathbb{G}(z')] = \log\left(\frac{1}{1-\overline{z}z'}\right).$$

- The variance of $\mathbb{G}$ goes to infinity when $|z| \to 1$, and for $z \in \mathbb{U}$, $\log X_n(z)$ does not converge in distribution.

- In a joint paper with Nikeghbali and Rouault, we prove that for *n* goes to infinity,

$$\sqrt{\frac{\beta}{2\log n}}\log X_n(z) \xrightarrow[n\to\infty]{} \mathcal{N}^{\mathbb{C}},$$

where $\mathcal{N}^{\mathbb{C}}$ denotes a complex Gaussian variable $Z$ such that

$$\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0,\ \mathbb{E}[|Z|^2] = 1.$$

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- Without normalization, $(\sqrt{\beta/2}\log X_n(z))_{z\in\mathbb{C}}$ tends in distribution to a complex Gaussian field on the unit circle, whose correlation between points $z, z' \in \mathbb{U}$ is given by $\log|z - z'|$. Note that this field is not defined on single points, since the correlation has a logarithmic singularity when $z'$ goes to $z$.

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- In this regularization, the correlation of the field saturates when $|z - z'|$ is of order $1/n$.
- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log\log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.

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- For this kind of regularization, it is conjectured that the maximum of the field is of order $\log n - (3/4) \log\log n$. This behavior (in particular the constant $-3/4$) is believed to be universal, i.e. not depending on the detail of the model.
- Such result has been proven for Gaussian regularizations (by Madaule, in 2015, then generalized by Ding, Roy and Zeitouni), for branching random walks and branching Brownian motion.

# Statement of the main result

- For $\beta = 2$, Fyodorov, Hiary and Keating (2012), have given a conjecture on the maximum of the characteristic polynomial, which is the following:

  $$\sup_{z\in\mathbb{U}} \log|X_n(z)| - \left(\log n - \frac{3}{4}\log\log n\right) \xrightarrow[n\to\infty]{} \frac{1}{2}(K_1 + K_2),$$

  in distribution, where $K_1$ and $K_2$ are two independent Gumbel random variables.

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- In November 2015, Arguin, Belius and Bourgade have proven that

$$\frac{\sup_{z\in\mathbb{U}} \log|X_n(z)|}{\log n} \xrightarrow[n\to\infty]{} 1$$

  in probability.

- In Feburary 2016, Paquette and Zeitouni have proven:

$$\frac{\sup_{z\in\mathbb{U}} \log|X_n(z)| - \log n}{\log\log n} \xrightarrow[n\to\infty]{} -\frac{3}{4}.$$

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- We expect that the conjecture of Fyodorov, Hiary and Keating can be generalized to β ensembles:

$$\sqrt{\beta/2}\sup_{z\in\mathbb{U}}\log|X_n(z)|-\left(\log n-\frac{3}{4}\log\log n\right)\xrightarrow[n\to\infty]{}K,$$

  where $K$ is a limiting random variable. It may be possible that $2K$ is the sum two independent Gumbel variables, but we have no argument supporting such a statement.

In a joint work with Chhaibi and Madaule (to be submitted soon...), we get the following result:

## Theorem

*For any function h tending to infinity at infinity,*

$$\left| \sqrt{\beta/2} \sup_{z \in \mathbb{U}} \Re \log X_n(z) - \left( \log n - \frac{3}{4} \log\log n \right) \right| \leq h(n),$$

$$\left| \sqrt{\beta/2} \sup_{z \in \mathbb{U}} \Im \log X_n(z) - \left( \log n - \frac{3}{4} \log\log n \right) \right| \leq h(n),$$

*with probability tending to* 1 *when n goes to infinity.*

The statement on the imaginary part gives information on the number of eigenvalues lying on arcs of the unit circle.

We deduce the following:

### Corollary

*For $z_1, z_2 \in \mathbb{U}$, let $N(z_1, z_2)$ be the number of points $\lambda_j$ lying on the arc coming counterclockwise from $z_1$ to $z_2$, and $N_0(z_1, z_2)$ its expectation (i.e. the length of the arc multiplied by $n/2\pi$). Then,*

$$\left| \pi\sqrt{\beta/8} \sup_{z_1, z_2 \in \mathbb{U}} |N(z_1, z_2) - N_0(z_1, z_2)| - \left(\log n - \frac{3}{4}\log\log n\right) \right| \leq h(n)$$

*with probability tending to* 1 *when n goes to infinity.*

- In the sequel of the talk, we sketch a proof of a weaker version of the upper bound:

## Theorem

*With probability tending to* 1*,*

$$\sqrt{\beta/2}\sup_{z\in\mathbb{U}}\Re\log X_n(z)\leq\log n-\frac{3}{4}\log\log n+\frac{3}{2}\log\log\log n+h(n),$$

$$\sqrt{\beta/2}\sup_{z\in\mathbb{U}}\Im\log X_n(z)\leq\log n-\frac{3}{4}\log\log n+\frac{3}{2}\log\log\log n+h(n).$$

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- At the end of the talk, we briefly give some elements of the proof of the lower bound.

# Orthogonal polynomials on the unit circle

If $\nu$ is a probability measure on the unit circle, the Gram-Schmidt procedure applied on $L^2(\nu)$ to the sequence $(z^k)_{k\geq 0}$ gives a sequence $(\Phi_k)_{0\leq k<m}$ of monic orthogonal polynomials, $m$ being the (finite or infinite) cardinality of the support of $\nu$. If $m<\infty$, the procedure stops after $\Phi_{m-1}$ since all $L^2(\nu)$ is spanned: we then define

$$\Phi_m(z) := \prod_{\lambda\in\mathrm{Supp}(\nu)} (z-\lambda),$$

which vanishes in $L^2(\nu)$. Moreover, we define $\Phi_k^*(z) := z^k\overline{\Phi_k(1/\bar{z})}$.

- There exists a sequence $(\alpha_j)_{0 \leq j < m}$ of complex numbers, $|\alpha_j| = 1$ if $j = m - 1 < \infty$, $|\alpha_j| < 1$ otherwise, called *Verblunsky coefficients*, such that the polynomials above satisfy the so-called *Szegö recursion*: for $j < m$,

$$\Phi_{j+1}(z) = z\Phi_j(z) - \overline{\alpha_j}\Phi_j^*(z),$$

$$\Phi_{j+1}^*(z) = -\alpha_j z\Phi_j(z) + \Phi_j^*(z).$$

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- Moreover, Killip and Nenciu have found an explicit probability distribution for the Verblunsky coefficients, for which one can recover the characteristic polynomial of the Circular Beta Ensemble.

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- Then, we have the equality in distribution:

$$X_n(z) = \Phi^*_{n-1}(z) - z\eta\Phi_{n-1}(z).$$

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- If we couple the polynomials in such a way that we have acutally an equality, then

$$\left(\sup_{z\in\mathbb{U}} \left| \log X_n(z) - \log \Phi^*_{n-1}(z) \right| \right)_{n\geq 1}$$

is tight: it is then sufficient to study the extreme values of $\log \Phi_n^*$ instead of $\log X_n$.

- The recursion can be rewritten by using the *deformed Verblunsky coefficients* $(\gamma_j)_{j\geq 0}$, which have the same modulii as $(\alpha_j)_{j\geq 0}$ and the same joint distribution.

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- We have, for $\theta \in [0, 2\pi)$,

$$\log \Phi_k^*(e^{i\theta}) = \sum_{j=0}^{k-1} \log\left(1 - \gamma_j e^{i\psi_j(\theta)}\right).$$

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- The so-called *relative Prüfer phases* $(\psi_k)_{k\geq 0}$ satisfy:

$$\psi_k(\theta) = (k+1)\theta - 2\sum_{j=0}^{k-1} \log \left(\frac{1-\gamma_j e^{i\psi_j(\theta)}}{1-\gamma_j}\right).$$

# Sketch of proof of a non-sharp upper bound

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- Indeed, if $\mathbb{U}_m$ denotes the set of $m$-th roots of unity, we have for all polynomials $Q$ of degree at most $n$:

$$\sup_{z \in \mathbb{U}} |Q(z)| \leq 14 \sup_{z \in \mathbb{U}_{2n}} |Q(z)|.$$

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- If $Q(0) = 1$ and $Q$ has all roots outside the unit disc, then

$$\sup_{z \in \mathbb{U}} \operatorname{Arg}(Q(z)) \leq \sup_{z \in \mathbb{U}_n} \operatorname{Arg}(Q(z)) + 2\pi.$$

- For any $z \in \mathbb{U}$, we have the equality in distribution:

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- From this result, we deduce an expression of $\log X_n(z)$ as a sum of independent random variables with explicit distributions:

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- For $\beta = 2$, this expression can be related to a splitting of a CUE matrix as a product of complex reflections (work by Bourgade, Hughes, Nikeghbali, Yor in 2008).
- In general, one can deduce an expression of the exponential moments of $\log X_n(z)$ as a product of ratios of Gamma functions, and then get the Central Limit theorem stated above.

- Moreover, we get for $s > 0, t \in \mathbb{R}$,

$$\mathbb{E}[e^{s\Re \log \Phi_k^*(z) + t\Im \log \Phi_k^*(z)}] \leq (ke)^{(s^2+t^2)/(2\beta)}.$$

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- Using a Chernoff bound with $s = \sqrt{2\beta}$, $t = 0$, we deduce that for $n \to \infty$,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}}\Re \log \Phi_n^*(z) \geq \log n + h(n)\right) = o(1/n)$$

and the same for the imaginary part.

- Using a union bound on the $2n$-th roots of unity,

$$\mathbb{P}\left(\sqrt{\frac{\beta}{2}}\sup_{z\in\mathbb{U}}\Re\log\Phi_n^*(z)\leq\log n+h(n)\right)\xrightarrow[n\to\infty]{}1,$$

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  which gives a weak version of the upper bound stated above.
- Moreover, if we define

  $$\mathcal{B}_n:=\{\lfloor e^j\rfloor,0\leq j\leq\lfloor\log n\rfloor\}\cup\{n\},$$

  then

  $$\mathbb{P}\left(\forall k\in\mathcal{B}_n,\sup_{z\in\mathbb{U}}\Re\log\Phi_k^*(z)\leq\log k+\log\log n+h(n)\right)\underset{n\to\infty}{\longrightarrow}1.$$

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- This estimate is useful in order to prove a sharper upper bound.

# Sketch of proof of a sharper upper bound

- In order to show the sharper (but not optimal) upper bound previously stated, it is sufficient to show:

$$\mathbb{P}\Bigg(\forall k \in \mathcal{B}_n, \sup_{z\in\mathbb{U}} \Re \log \Phi_k^*(z) \leq \log k + \log\log n + h(n),$$

$$\sup_{z\in\mathbb{U}} \Re \log \Phi_n^*(z) \geq \log n - \frac{3}{4}\log\log n + \frac{3}{2}\log\log\log n + h(n)\Bigg) \xrightarrow[n\to\infty]{} 0.$$

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- By doing a union bound on $\mathbb{U}_{2n}$, it is sufficient to prove that the probability of the same event for a single $z \in \mathbb{U}$ is $o(1/n)$ when $n$ goes to infinity.

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- We have an equality in law:

$$\log(1-\gamma_k) = \log\left(1 - e^{i\Theta_k}\sqrt{\frac{E_k}{E_k+\Gamma_k}}\right)$$

where $(E_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$, $(\Theta_k)_{k\geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k+1)$ and uniform on $[0, 2\pi)$.

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where $(E_k)_{k\geq 0}$, $(\Gamma_k)_{k\geq 0}$, $(\Theta_k)_{k\geq 0}$ are independent variables, respectively exponentially distributed, Gamma of parameter $(\beta/2)(k+1)$ and uniform on $[0, 2\pi)$.

- If we replace $E_k + \Gamma_k$ by $(\beta(k+1))/2$ and $\log(1-y)$ by $-y$, we get a Gaussian variable of variance $1/(\beta(k+1))$.

- One can prove that $(\sqrt{\beta/2}\log\Phi_k^*(z))_{k\geq 0, z\in\mathbb{U}}$ can be coupled, with an a.s. bounded difference, with a field $\big(Z_k(z)\big)_{k\geq 0, z\in\mathbb{U}}$, with complex Gaussian marginals, with independent increments for fixed $\theta$:

$$Z_k(e^{i\theta}) := \sum_{j=0}^{k-1} \frac{\mathcal{N}_j^{\mathbb{C}} e^{i\psi_j(\theta)}}{\sqrt{j+1}}.$$

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- In this way, we can deduce that it is essentially sufficient to show ($N$ corresponding to $\log n$), for a Brownian motion $W$ that:

$$\mathbb{P}\Big(\forall j \in \{1,2,\dots,N-1\}, W_j \leq \sqrt{2}\,(j+\log N + h(N)),$$

$$W_N \geq \sqrt{2}\left(N - \frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\Big) = o(e^{-N}).$$

- Using Girsanov's theorem, it is enough to show

$$\mathbb{P}\left(\forall j \in \{1,2,\ldots,N-1\}, W_j \leq \sqrt{2}\left(\log N + h(N)\right),\right.$$

$$\left. W_N \geq \sqrt{2}\left(-\frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\right) = \mathcal{O}\left(N^{-3/2}(\log N)^3\right).$$

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- This result can be deduced from a suitable version of the ballot theorem, or from the joint law of a Brownian motion and its past supremum.

- Using Girsanov's theorem, it is enough to show

$$\mathbb{P}\left(\forall j \in \{1,2,\ldots,N-1\}, W_j \leq \sqrt{2}\,(\log N + h(N)),\right.$$

$$\left. W_N \geq \sqrt{2}\left(-\frac{3}{4}\log N + \frac{3}{2}\log\log N + h(N)\right)\right) = \mathcal{O}\left(N^{-3/2}(\log N)^3\right).$$

- This result can be deduced from a suitable version of the ballot theorem, or from the joint law of a Brownian motion and its past supremum.
- We can remove the term $(3/2)\log\log\log n$ in the main result by slightly deforming the barrier $\log k + \log\log n + h(n)$, and by using a sufficiently powerful version of the ballot theorem.

# Strategy for the lower bound

- In order to get the lower bound, we have to show that with high probability, there exists $\theta \in [0, 2\pi)$ such that

$$\Re Z_n(e^{i\theta}) \geq \log n - \frac{3}{4} \log \log n - h(n).$$

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$$\Re Z_n(e^{i\theta}) \geq \log n - \frac{3}{4} \log\log n - h(n).$$
- Let $E_n(\theta)$ be any event implying the previous inequality. It is sufficient to show:
$$\mathbb{P}(N_n > 0) \xrightarrow[n\to\infty]{} 1,$$
where $N_n$ is the number of $j \in \{0, \dots, n-1\}$ such that $E_n(e^{2i\pi j/n})$ occurs.

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- For that, we need to choose events $E_n(\theta)$, in such a way that their probability is not too small and that $E_n(\theta)$ and $E_n(\theta')$ are not too much correlated if $\theta$ is not too close to $\theta$.

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- Since the Prüfer phases incrase by approximately $\theta$ at each step, for $\theta \in [0,\pi]$, the increments of the random walks $(Z_k(0))_{k\in\mathcal{B}_n}$ and $(Z_k(\theta))_{k\in\mathcal{B}_n}$ are "roughly similar" for $k \leq 1/\theta$ and "roughly independent" afterwards.
- We can then do similar computations as for branching Gaussian random walks.

Thank you for your attention!