Strong convergence for the CUE

Ashkan Nikeghbali
University of Zurich

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The random matrix model

CUE: The Circular Unitary Ensemble

- The unitary group $U(n)$ with the Haar measure;
- Eigenvalues on the unit circle; $e^{i\theta_1}, \cdots, e^{i\theta_n}$.
- Weyl's integration formula: the joint density of the eigenangles $(\theta_1, \cdots, \theta_n) \in [0, 2\pi]^n$ is:

$$\frac{1}{(2\pi)^n n!} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

The microscopic scale

We consider

$$Z_n(X) = \det\left(\mathrm{Id} - U_n^{-1}X\right) = \det\left(\mathrm{Id} - U_n^{*}X\right).$$

(i) Is there a random analytic function arising from the characteristic polynomial in the n-limit?

(ii) What is the n-limit of

$$R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r) := \frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})};$$

(iii) Same question for the logarithmic derivative;

The ratios

- Expectations for ratios of the form

$$\frac{Z_n(e^{2i\alpha_1\pi})\cdots Z_n(e^{2i\alpha_r\pi})}{Z_n(e^{2i\beta_1\pi})\cdots Z_n(e^{2i\beta_r\pi})}$$

  have been extensively studied by Fyodorov-Strahov, Borodin-Olshanski-Strahov, Bump-Gamburd, Conrey-Farmer-Zirnbauer, Conrey-Snaith, etc.

- Same thing for the logarithmic derivative (Conrey-Snaith, Conrey-Farmer-Zirnbauer, Farmer-Gonek-Montgomery)

$$\frac{e^{2i\alpha_1\pi}Z_n'(e^{2i\alpha_1\pi})}{Z_n(e^{2i\alpha_1\pi})}\cdots\frac{e^{2i\alpha_k\pi}Z_n'(e^{2i\alpha_k\pi})}{Z_n(e^{2i\alpha_k\pi})}.$$

Determinantal structure

- If u_n is distributed according to Haar measure, then one can define, for $1 \leq p \leq n$, the p-point correlation function $\rho_p^{(n)}$ of the eigenangles, as follows: for any bounded, measurable function ϕ from $\mathbb{R}^p$ to $\mathbb{R}$,

$$\mathbb{E}\left[\sum_{1\leq j_1 \neq \cdots \neq j_p \leq n} \phi(\theta_{j_1}^{(n)}, \ldots, \theta_{j_p}^{(n)})\right] = \int_{[0,2\pi)^p} \rho_p^{(n)}(t_1, \ldots, t_p)\phi(t_1, \ldots, t_p)dt_1 \ldots dt_p.$$

- If the kernel $K^{(n)}$ is defined by

$$K^{(n)}(t) := \frac{\sin(nt/2)}{2\pi \sin(t/2)}$$

then the p-point correlation function is be given by

$$\rho_p^{(n)}(t_1, ..., t_n) = \det\left(K^{(n)}(t_j - t_k)\right)_{j,k=1}^{p}.$$

Proposition

Let E_n denote the set of eigenvalues taken in $(-\pi, \pi]$ and multiplied by $n/2\pi$. Let Define for $y \neq y^{'}$

$$K^{(\infty)}(y, y^{'}) = \frac{\sin[\pi(y^{'} - y)]}{\pi(y^{'} - y)}$$

and

$$K^{(\infty)}(y, y) = 1.$$

Then there exists a point process E_∞ such that for all $r \in \{1, \dots, n\}$, and for all measurable and bounded functions F with compact support from $\mathbb{R}^r$ to $\mathbb{R}$:

$$\mathbb{E}\left(\sum_{x_1 \neq \cdots \neq x_r \in E_n} F(x_1, \dots, x_r)\right) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^r} F(y_1, \dots, y_r)\rho_r^{(\infty)}(y_1, \dots, y_r)dy_1 \dots dy_r,$$

where

$$\rho_r^{(\infty)}(y_1, \dots, y_r) = \det((K^{(\infty)}(y_j, y_k))_{1\leq j,k\leq r}).$$

Coupling all dimensions

- The idea finds its roots in the construction of virtual permutations by Kerov, Olshanski, Vershik.
- Using a coupling by Neretin, Borodin-Olshanski had already obtained the a.s. convergence of the eigenvalues.
- We propose an alternative probabilistic construction which contains both constructions of virtual transformations and which provides a.s. convergence for both eigenvalues and eigenvectors with a good control on error terms.

Complex Reflections

- We endow $\mathbb{C}^n$ with the scalar product: $\langle x, y\rangle = \sum_{k=1}^n x_k \bar{y}_k$.
- A reflection is a unitary transformation such that r such that it is the identity or the rank of $Id - r$ is 1.
- Every reflection can be represented as:

$$r(x) = x - (1-\alpha)\frac{\langle x, a\rangle}{\langle a, a\rangle}a,$$

  where a is some vector and α is an element of the unit circle.
- Given two distinct unit vectors e and m , there exists a unique complex reflection r such that $r(e) = m$ and it is given by

$$r(x) = x - \frac{\langle x, m-e\rangle}{1-\langle e, m\rangle}(m-e).$$

Constructing virtual isometries $(u_n)_{n\geq 1}$

The sequence $(u_n)_{n\geq 1}$ can be constructed in the following way (BNN):

1. One considers a sequence $(x_n)_{x\geq 1}$ of independent random vectors, x_n being uniform on the unit sphere of $\mathbb{C}^n$.
2. Almost surely, for all $n \geq 1$, x_n is different from the last basis vector e_n of $\mathbb{C}^n$, which implies that there exists a unique complex reflection $r_n \in U(n)$ such that $r_n(e_n) = x_n$ and $I_n - r_n$ has rank one.
3. We define $(u_n)_{n\geq 1}$ by induction as follows: $u_1 = x_1$ and for all $n \geq 2$,

$$u_n = r_n \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

4. We note $U^\infty := \{(u_n)_{n\geq 1}\}$ the space of virtual isometries. We define on U^∞ the measure μ_∞ as the projective limite of the μ_n's.

A probabilistic approach to the Keating-Snaith formula

- The following identity in law holds (BHNY):

$$\det(I - U) = \prod_{k=1}^{n} \left(1 + e^{i\theta_k} \sqrt{\beta_{1,k-1}}\right)$$

  where are random variables in sight are independent (θ_k's are uniform on $(0, 2\pi)$ and $\beta_{1,k-1}$ is a beta r.v. with parameters 1 and $k-1$).
- The weakness of these identities in law is that it seems very hard to say anything about the characteristic polynomial at evaluated at two or more points. In particular there is not much hope to build infinite dimensional objects (i.e. random analytic functions or random operators).
- This splitting can be extended to the circular beta ensemble or even to the Jacobi circular ensemble (through deformed Verblunsky coeffcients).

Convergence of eigenangles

Theorem

(i) (BNN, MNN) There is a sine-kernel point process $(y_k)_{k\in\mathbb{Z}}$ such that almost surely,

$$y_k^{(n)} \equiv \frac{n}{2\pi}\theta_k^{(n)} = y_k + O((1+k^2)n^{-\frac{1}{3}+\epsilon}),$$

for all $n \geq 1$, $|k| \leq n^{1/4}$ and $\epsilon > 0$, where the implied constant may depend on $(u_m)_{m\geq 1}$ and ϵ, but not on n and k.

(ii) (CNN) Almost surely, and uniformly in k and n:

$$y_k^{(n)} \equiv \frac{n}{2\pi}\theta_k^{(n)} = k + O\left(\log(2+|k|)\right).$$

Theorem (CNN)

Define

$$\xi_n(z) = \frac{Z_n(e^{2iz\pi/n})}{Z_n(1)}.$$

Almost surely and uniformly on compact subsets of $\mathbb{C}$, we have the convergence:

$$\xi_n(z) \overset{n\to\infty}{\longrightarrow} \xi_\infty(z) := e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right)$$

Here, the infinite product is not absolutely convergent. It has to be understood as the limit of the following product, obtained by regrouping the factors two by two:

$$\left(1 - \frac{z}{y_0}\right) \prod_{k\geq 1} \left[\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right],$$

which is absolutely convergent.

Remarks

- Functional Equation:

$$\xi_n(z) = \frac{Z_n(e^{2iz\pi/n})}{Z_n(1)}.$$

- We have the following representation for the characteristic polynomial within $1/n$ distance of the unit circle:

$$Z_n(e^{2iz\pi/n}) = Z_n(1) \times \xi_n(z).$$

- We have a.s. as $n \to \infty$,

$$\frac{2i\pi}{n} \frac{Z_n'(e^{2i\pi z/n})}{Z_n(1)} \to \xi_\infty'.$$

- Many new non trivial limit theorems follow from this strong convergence (e.g. limit theorems à la Hejhal):

$$\frac{1}{\sqrt{1/2 \log n}} \left(\log Z_n'(1) - \log \log n, \log Z_n(1)\right) \to (\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}).$$

The steps in the proof

- We first observe that

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right).$$

- Then for any $A \geq 2$, and $z \in K$, K a compact set, one has:

$$\left| \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k}\right) \right| \leq$$

$$\left| \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{|k| \leq A} \left(1 - \frac{z}{y_k}\right) \right| + O_K\left(\frac{\log A}{A}\right)$$

- Then use $y_k^{(n)} \to y_k$ almost surely.

Ratios

Proposition (CNN)

Let $r \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$ but $\beta_j \notin (y_k)_{k \in \mathbb{Z}}$, for all $1 \le j \le r$. Then the following convergence holds a.s. as $n \to \infty$:

$$\frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})} \to \frac{\xi_\infty(\alpha_1) \cdots \xi_\infty(\alpha_r)}{\xi_\infty(\beta_1) \cdots \xi_\infty(\beta_r)}$$

The number theory connection

Conjecture

Let ω be a uniform random variable on $[0,1]$ and $T>0$ a real parameter going to infinity. We conjecture the following convergence in law, uniformly in the parameter z on every compact set:

$$\left(\frac{\zeta\left(\frac{1}{2}+i\omega T-\frac{i2\pi z}{\log T}\right)}{\zeta\left(\frac{1}{2}+i\omega T\right)};z\in\mathbb{C}\right)\overset{T\to\infty}{\longrightarrow}(\xi_\infty(z);z\in\mathbb{C})$$

By taking logarithmic derivatives, it is natural also to conjecture the following convergence

$$\left(\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T-\frac{i2\pi z}{\log T}\right);z\in\mathbb{C}\right)\overset{T\to\infty}{\longrightarrow}\left(\frac{\xi'_\infty}{\xi_\infty}(z);z\in\mathbb{C}\right)$$

on compact sets bounded away from the real line.

Proposition

We have, for $z \notin \mathbb{R}$,

$$\frac{\xi_\infty'}{\xi_\infty}(z) = i\pi + \sum_{k\in\mathbb{Z}} \frac{1}{z - y_k} =: i\pi + \frac{1}{z - y_0} + \sum_{k=1}^{\infty} \left(\frac{1}{z - y_k} + \frac{1}{z - y_{-k}} \right),$$

and when the random variable U is fixed:

$$\frac{-i2\pi}{\log T} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + iTU - \frac{i2\pi z}{\log T} \right) = i\pi + \sum_{\tilde{\gamma}} \frac{1}{z - \tilde{\gamma}} + o(1)$$

where

$$\tilde{\gamma} := \frac{-\log T}{2\pi i} \left(\rho - \frac{1}{2} - iUT \right)$$

with ρ a zero of ζ. The infinite sum on $\tilde{\gamma}$ has to be understood as follows:

$$\sum_{\tilde{\gamma}} \frac{1}{z - \tilde{\gamma}} = \frac{1}{z - \tilde{\gamma}_0} + \sum_{k=1}^{\infty} \left(\frac{1}{z - \tilde{\gamma}_k} + \frac{1}{z - \tilde{\gamma}_{-k}} \right),$$

where $(\tilde{\gamma}_k)_{k\in\mathbb{Z}}$ are ordered by increasing real part, increasing imaginary part if they have the same real part, and counted with multiplicity.

Moments of the logarithmic derivative

Proposition

Almost surely, for all $z \notin \{y_k, k \in \mathbb{Z}\}$,

$$\frac{\xi'_\infty(z)}{\xi_\infty(z)} = i\pi + \lim_{A \to \infty} \sum_{[y_k|<A} \frac{1}{z - y_k}.$$

Remark

We also have

$$\frac{\xi'_n(z)}{\xi_n(z)} = i\pi + \lim_{A \to \infty} \sum_{[y_k^{(n)}|<A} \frac{1}{z - y_k^{(n)}}.$$

From now, we will allow n to be either ∞ or a strictly positive integer, and we will write by convention $y_k^{(\infty)} := y_k$. Moreover, we define:

$$\sum_{|y_k^{(n)}|>A} \frac{1}{z - y_k^{(n)}} := \frac{\xi'_n(z)}{\xi_n(z)} - i\pi - \sum_{[y_k^{(n)}|\leq A} \frac{1}{z - y_k^{(n)}}.$$

Proposition (CNN)

Let $K \subset \mathbb{C} \backslash \mathbb{R}$ be a compact set. Then, there exists $C_K > 0$, depending only on K, such that for all $p \geq 0$ and for all $A \geq C_K(1 + p^2 \log(2+p))$,

$$\sup_{n \in \mathbb{N} \sqcup \{\infty\}} \mathbb{E}\left[\sup_{z \in K} e^{p\left|\sum_{|y_k^{(n)}|>A} \frac{1}{z - y_k^{(n)}}\right|}\right] \leq 1 + \frac{C_K p \log A}{\sqrt{A}}.$$

Corollary

For any compact set K of $\mathbb{C} \backslash \mathbb{R}$, and for all $p \geq 1$, there exists an absolute constant $C_{p,K}$ such that:

$$\forall A \geq 0, \sup_{z \in K} \mathbb{E}\left(\left|\sum_{|y_k|>A} \frac{1}{z - y_k}\right|^p\right)^{\frac{1}{p}} \leq C_{p,K} \frac{\log(2+A)}{\sqrt{1+A}}.$$

Moments of the logarithmic derivative

- For fixed $z_1, z_2, \dots, z_p \notin \mathbb{R}$,

$$\forall p \geq 1, \frac{\xi_\infty'}{\xi_\infty}(z_1) \dots \frac{\xi_\infty'}{\xi_\infty}(z_p) \in L^p,$$

- $$\mathbb{E}\left(\frac{\xi_\infty'}{\xi_\infty}(z_1) \dots \frac{\xi_\infty'}{\xi_\infty}(z_p)\right) = \lim_{A \to \infty} \mathbb{E}\left(\prod_{j=1}^{p}\left(i\pi + \sum_{|y_k| < A} \frac{1}{z_j - y_k}\right)\right).$$

- The last quantity can be computed thanks to the sine kernel correlation functions of order less or equal than p, on the segment $[-A, A]$.

First moment

$M_1(z), z \notin \mathbb{R}$:

$$\begin{aligned} M_1(z) &:= \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\right) \\ &= i\pi + \lim_{A\to\infty} \mathbb{E}\left(\sum_{|y_k|\leq A} \frac{1}{z-y_k}\right) \\ &= i\pi + \lim_{A\to\infty} \int_{[-A,A]} dy \, \frac{\rho_1(y)}{z-y} \\ &= i\pi \left(1 - \operatorname{sgn}\left(\mathfrak{Im}(z)\right)\right) \\ &= i2\pi \mathbf{1}_{\{\mathfrak{Im}(z)<0\}} \end{aligned}$$

Second moment

$$M_2(z,z') = \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\frac{\xi'_\infty}{\xi_\infty}(z')\right).$$

We have

$$M_2(z,z') = -4\pi^2\mathbf{1}_{\mathfrak{Im}(z)<0,\mathfrak{Im}(z')<0} - \frac{1-e^{2i\pi(z-z')\,\mathrm{sgn}(\mathfrak{Im}(z-z'))}}{(z-z')^2}\mathbf{1}_{\mathfrak{Im}(z)\mathfrak{Im}(z')<0}.$$

$$\tilde{M}_2(z,z') := \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\overline{\frac{\xi'_\infty}{\xi_\infty}(z')}\right).$$

We have:

$$\tilde{M}_2(z,z') = 4\pi^2\mathbf{1}_{\mathfrak{Im}(z)<0,\mathfrak{Im}(z')<0} - \frac{1-e^{2i\pi(z-\overline{z'})\,\mathrm{sgn}(\mathfrak{Im}(z-\overline{z'}))}}{(z-\overline{z'})^2}\mathbf{1}_{\mathfrak{Im}(z)\mathfrak{Im}(z')>0}.$$

Conjecture

In particular:

$$\mathbb{E}\left(\left|\frac{\xi_\infty'}{\xi_\infty}(z)\right|^2\right) = 4\pi^2 \mathbf{1}_{\mathfrak{Im}(z)<0} + \frac{1-e^{-4\pi|\mathfrak{Im}(z)|}}{4\mathfrak{Im}^2(z)}.$$

Conjecture

$$\lim_{T\to\infty} \frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a}{\log T}\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a'}{\log T}\right)\right)$$

$$= \mathbf{1}_{\mathfrak{Re}(a)<0,\mathfrak{Re}(a')<0} - \frac{1-e^{-\left(a'-a\right)\operatorname{sgn}\mathfrak{Re}(a'-a)}}{(a-a')^2}\mathbf{1}_{\mathfrak{Re}(a)\mathfrak{Re}(a')<0}$$

$$\lim_{T\to\infty} \frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a}{\log T}\right)\overline{\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a'}{\log T}\right)}\right)$$

$$= \mathbf{1}_{\mathfrak{Re}(a)<0,\mathfrak{Re}(a')<0} + \frac{1-e^{-\left(a+\overline{a'}\right)\operatorname{sgn}\mathfrak{Re}(a+\overline{a'})}}{\left(a+\overline{a'}\right)^2}\mathbf{1}_{\mathfrak{Re}(a)\mathfrak{Re}(a')>0}$$

Moments of the ratios

Proposition

For $z_1, \dots, z_k, z'_1, \dots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, and for all $n \in \mathbb{N} \sqcup \{\infty\}$,

$$\mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} \right) < \infty$$

Moreover, for every compact set K in $\mathbb{C} \backslash \mathbb{R}$, we have the following convergence, uniformly in $z_1, z_2, \dots, z_k, z'_1, \dots, z'_k \in K$:

$$\mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} \right) \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right).$$

Theorem

For $(z_1, \ldots, z_k) \in (\mathbb{C}\backslash\mathbb{R})^k$ and $(z'_1, \ldots, z'_k) \in \mathbb{C}^k$, such that for $1 \leq i, j \leq k$, $z_i - z'_j$ is not an integer multiple of n,

$$\det\left(\frac{1}{e^{\frac{i2\pi z_i}{n}} - e^{\frac{i2\pi z'_j}{n}}}\right)_{i,j=1}^{k} \mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)}\right) = \det\left(\frac{1}{e^{\frac{i2\pi z_i}{n}} - e^{\frac{i2\pi z'_j}{n}}} \mathbb{E}\left(\frac{\xi_n(z'_j)}{\xi_n(z_i)}\right)\right)_{i,j=1}^{k}$$

and moreover:

$$\mathbb{E}\left(\frac{\xi_n(z')}{\xi_n(z)}\right) = \begin{cases} 1 & \text{if } \mathfrak{Im}(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \mathfrak{Im}(z) < 0 \end{cases}$$

Ratios Formula

Theorem

For all $z_1, \dots, z_k, z'_1, \dots, z'_k \in \mathbb{C} \backslash \mathbb{R}$ such that $z_i \neq z'_j$ for $1 \leq i, j \leq n$, we have

$$\det\left(\frac{1}{z_i - z'_j}\right)_{i,j=1}^{k} \mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right) = \det\left(\frac{1}{z_i - z'_j} \mathbb{E}\left(\frac{\xi_\infty(z'_j)}{\xi_\infty(z_i)}\right)\right)_{i,j=1}^{k}$$

and moreover:

$$\mathbb{E}\left(\frac{\xi_\infty(z')}{\xi_\infty(z)}\right) = \begin{cases} 1 & \text{if } \mathfrak{Im}(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \mathfrak{Im}(z) < 0 \end{cases}$$

Example

We note that

$$\overline{\xi_\infty(z)} = e^{-2i\pi\bar{z}}\xi_\infty(\bar{z}).$$

We get for all $z, z' \notin \mathbb{R}$,

$$\mathbb{E}\left[\left|\frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right] = e^{-4\pi\mathfrak{Im}(z'-z)\mathbf{1}_{\mathfrak{Im}(z)<0}}\left(1 + (1 - e^{-4\pi\mathfrak{Im}(z')\,\mathbf{sgn}(\mathfrak{Im}(z))})\frac{|z-z'|^2}{4\mathfrak{Im}(z)\mathfrak{Im}(z')}\right).$$

Conjecture

Let ω be a uniform random variable on $[0,1]$ and $T > 0$ a real parameter going to infinity. Then, for all $z_1, \dots, z_k, z'_1, \dots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, such that $z_i \neq z'_j$ for all i, j,

$$\mathbb{E}\left(\prod_{j=1}^{k} \frac{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z'_j}{\log T}\right)}{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z_j}{\log T}\right)}\right)$$

$$\underset{T \to \infty}{\longrightarrow} \det\left(\frac{1}{z_i - z'_j}\right)^{-1} \det\left(\frac{\mathbf{1}_{\mathfrak{Im}(z_i)>0} + e^{2i\pi(z'_j - z_i)}\mathbf{1}_{\mathfrak{Im}(z_i)<0}}{z_i - z'_j}\right)_{i,j=1}^{k},$$

where the last expression is well-defined where the z_i and the z'_j are all distinct, and is extended by continuity to the case where some of the z_i or some of the z'_j are equal.

Fluctuations for the logarithmic derivative

Proposition

For $z \in \mathbb{C} \backslash \mathbb{R}$, let

$$F(z) := \frac{\xi'_\infty(z)}{\xi_\infty(z)} - 2i\pi \mathbf{1}_{\mathfrak{Im} z < 0}.$$

Then, one has the convergence in law:

$$(LF(Lz))_{z \in \mathbb{C} \backslash \mathbb{R}} \underset{L \to \infty}{\longrightarrow} (G(z))_{z \in \mathbb{C} \backslash \mathbb{R}},$$

where $(G(z))_{z \in \mathbb{C} \backslash \mathbb{R}}$ is a centred gaussian process, which admits a holomorphic version, with covariance structure given, for all $z_1, z_2 \notin \mathbb{R}$, by

$$\mathbb{E}[G(z_1)G(z_2)] = -\frac{\mathbf{1}_{\mathfrak{Im}(z_1)\mathfrak{Im}(z_2) < 0}}{(z_2 - z_1)^2},$$

$$\mathbb{E}[G(z_1)\overline{G(z_2)}] = -\frac{\mathbf{1}_{\mathfrak{Im}(z_1)\mathfrak{Im}(z_2) > 0}}{(\overline{z_2} - z_1)^2}.$$