# Theory of Barnes Beta Probability Distributions

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Based on: IMRN '12, ECP '13, Forum Math '14, DO '16

# Overview and Motivating Applications

- A new family of laws $\beta_{M,N}(a,\, b) > 0,\ M \leq N+1,\ M, N \in \mathbb{N}$, $a = (a_1 \cdots a_M) > 0,\ b = (b_0, b_1 \cdots b_N) > 0$, such that $\mathbf{E}[\beta_{M,N}(a,\, b)^q]$ is a product of ratios of $M-$gamma factors, $\log \beta_{M,N}(a,\, b)$ is infinitely divisible

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- $\beta_{1,0},\, \beta_{2,2}$ : 1) Selberg and Morris integral probability distributions, 2) Maximum distribution of gaussian free field (GFF) on the unit interval/circle in the freezing framework, 3) Mod-gaussian convergence of exponential functionals of mesoscopic $\mathcal{H}^{1/2}(\mathbb{R})$ statistics (Nonlinearity '16)

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- $\beta_{2M,3M}$ as $M \to \infty$ : Riemann xi function (Forum Math '14)

# Preliminaries

- Barnes multiple gamma function: $\Gamma_M(w \,|\, a)$, $a = (a_1 \cdots a_M) > 0$

$$\Gamma_M^{-1}(w \,|\, a) = e^{P(w \,|\, a)}\, w \prod_{n_1,\cdots,n_M=0}^{\infty}{}' \Big(1+\frac{w}{\Omega}\Big) \exp\Big(\sum_{k=1}^{M} \frac{(-1)^k}{k} \frac{w^k}{\Omega^k}\Big),$$

for $\mathrm{Re}(w) > 0$, where $\Omega \triangleq \sum_{i=1}^{M} n_i a_i$, $a_i > 0$, continued analytically to $w \in \mathbb{C}$ with poles at $s = -\sum_{i=1}^{M} n_i\, a_i$

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- Functional equation: let $\hat{a}_i = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_M)$,

$$\Gamma_M(w \,|\, a) = \Gamma_{M-1}(w \,|\, \hat{a}_i)\, \Gamma_M\big(w + a_i \,|\, a\big),\ \Gamma_0(w) = 1/w$$

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- Modern theory is due to Ruijsenaars (2000)

# Definition of Barnes Beta Distributions

- Given $a = (a_1 \cdots a_M) > 0,\ b = (b_0, b_1 \cdots b_N) > 0$, let

$$\eta_{M,N}(q\,|a,b) = \frac{\Gamma_M(q+b_0|a)}{\Gamma_M(b_0|a)} \prod_{j_1=1}^{N} \frac{\Gamma_M(b_0+b_{j_1}|a)}{\Gamma_M(q+b_0+b_{j_1}|a)} \times$$

$$\times \prod_{j_1<j_2}^{N} \frac{\Gamma_M(q+b_0+b_{j_1}+b_{j_2}|a)}{\Gamma_M(b_0+b_{j_1}+b_{j_2}|a)} \prod_{j_1<j_2<j_3}^{N} \frac{\Gamma_M(b_0+b_{j_1}+b_{j_2}+b_{j_3}|a)}{\Gamma_M(q+b_0+b_{j_1}+b_{j_2}+b_{j_3}|a)} \cdots$$

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- Example: $M = N = 2,\ a = (a_1, a_2)$,

$$\eta_{2,2}(q\,|\,a,b) = \frac{\Gamma_2(q+b_0|a)}{\Gamma_2(b_0|a)} \frac{\Gamma_2(b_0+b_1|a)}{\Gamma_2(q+b_0+b_1|a)} \frac{\Gamma_2(b_0+b_2|a)}{\Gamma_2(q+b_0+b_2|a)} \times$$

$$\times \frac{\Gamma_2(q+b_0+b_1+b_2|a)}{\Gamma_2(b_0+b_1+b_2|a)}$$

# Existence of Barnes Beta Distributions

**Theorem (ECP '13, Forum Math '14, DO '16)**

*1) Let* $M \le N+1$, $\mathrm{Re}(q) > -b_0$, *then* $\eta_{M,N}(q\,|a, b)$ *is the Mellin transform of a probability distribution* $\beta_{M,N}(a, b)$ *on* $(0, 1]$ *if* $M \le N$ *and on* $(0, \infty)$ *if* $M = N+1$

$$\mathbf{E}\big[\beta_{M,N}(a, b)^q\big] = \eta_{M,N}(q \,|\, a, b).$$

*2)* $\log \beta_{M,N}(a, b)$ *is infinitely divisible. If* $M \le N$,

$$\mathbf{E}\Big[e^{q\log \beta_{M,N}(a,b)}\Big] = \exp\Big(\int_0^\infty (e^{-tq}-1)e^{-b_0 t}\frac{\prod_{j=1}^N (1-e^{-b_j t})}{\prod_{i=1}^M (1-e^{-a_i t})}\frac{dt}{t}\Big).$$

*3)* $\beta_{M,N}(a, b)$ *has density iff* $M = N$ *or* $M = N+1$.

*4) Let* $\kappa > 0$, $M \le N$, *then* $\beta^{\kappa}_{M,N}(\kappa a, \kappa b) \overset{\text{in law}}{=} \beta_{M,N}(a,\, b)$

# Example of $\beta_{2,2}(\tau, b)$, $(a_1 = 1, a_2 = \tau)$

- Functional equation,

$$\mathbf{E}\left[\beta_{2,2}(\tau,b)^{q+1}\right] = \mathbf{E}\left[\beta_{2,2}(\tau,b)^{q}\right] \frac{\Gamma\left(\frac{(q+b_0+b_1)}{\tau}\right)\Gamma\left(\frac{(q+b_0+b_2)}{\tau}\right)}{\Gamma\left(\frac{(q+b_0)}{\tau}\right)\Gamma\left(\frac{(q+b_0+b_1+b_2)}{\tau}\right)}$$

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- Moments, let $k \in \mathbb{N}$,

$$\mathbf{E}\left[\beta_{2,2}(\tau, b)^{k}\right] = \prod_{l=0}^{k-1}\left[\frac{\Gamma\left(\frac{(l+b_0+b_1)}{\tau}\right)\Gamma\left(\frac{(l+b_0+b_2)}{\tau}\right)}{\Gamma\left(\frac{(l+b_0)}{\tau}\right)\Gamma\left(\frac{(l+b_0+b_1+b_2)}{\tau}\right)}\right]$$

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- Shintani factorization,

$$\mathbf{E}\left[\beta_{2,2}(\tau,b)^{q}\right] = \prod_{k=0}^{\infty}\left[\frac{\Gamma\left(\frac{(q+k+b_0)}{\tau}\right)}{\Gamma\left(\frac{(k+b_0)}{\tau}\right)}\frac{\Gamma\left(\frac{(k+b_0+b_1)}{\tau}\right)}{\Gamma\left(\frac{(q+k+b_0+b_1)}{\tau}\right)}\frac{\Gamma\left(\frac{(k+b_0+b_2)}{\tau}\right)}{\Gamma\left(\frac{(q+k+b_0+b_2)}{\tau}\right)}\star\right.$$
$$\left.\star\frac{\Gamma\left(\frac{(q+k+b_0+b_1+b_2)}{\tau}\right)}{\Gamma\left(\frac{(k+b_0+b_1+b_2)}{\tau}\right)}\right]$$

# GFF and Multiplicative Chaos on the Unit Interval

- GFF on the unit interval: $V_\varepsilon(u)$, $u \in [0,\, 1]$, centered gaussian,

$$\mathbf{Cov}\left[V_\varepsilon(u),\; V_\varepsilon(v)\right] = \begin{cases} -\,2\,\log|u-v|,\, \varepsilon \ll |u-v| \le 1, \\ 2\left(\kappa - \log\varepsilon\right),\; u=v,\;\; \kappa \ge 0. \end{cases}$$

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- Mandelbrot '72, Kahane '85, Bacry-Muzy '01: let $0 \leq \beta < 1$,

$$e^{-\beta^2(\kappa - \log\varepsilon)} \int_a^b e^{\beta V_\varepsilon(u)}\,du \to M_\beta(a,b),\ \varepsilon \to 0,$$

$$\mathbf{E}[M_\beta(a,b)] = |b-a|,\ M_\beta(a,a+\tau) \overset{\text{in law}}{=} M_\beta(0,\tau)$$

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- Moments are given by the Selberg integral: let $n < 1/\beta^2$,

$$\mathbf{E}\left[\left(\int_{[0,1]} s^{\lambda_1}(1-s)^{\lambda_2}\, M_\beta(ds)\right)^n\right] = \int_{[0,1]^n} \prod_{i=1}^n s_i^{\lambda_1}(1-s_i)^{\lambda_2} \times$$

$$\times \prod_{i<j} |s_i - s_j|^{-2\beta^2} ds_1 \cdots ds_n$$

# Multiplicative Chaos on the Interval and Barnes Beta I

**Theorem (IMRN '12, Forum Math '14)**

*Let* $\tau = 1/\beta^2 > 1$, *define* $L = \exp\big(\mathcal{N}(0,\, 4\log 2/\tau)\big)$,

$$X_1 = \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau+\tau\lambda_1,\ b_1 = b_2 = \tau(\lambda_2-\lambda_1)/2\Big),$$

$$X_2 = \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau+\tau(\lambda_1+\lambda_2)/2,\ b_1 = 1/2,\ b_2 = \tau/2\Big),$$

$$X_3 = \beta_{2,2}^{-1}\Big(\tau, b_0 = 1+\tau,\ b_1 = b_2 = (1+\tau+\tau\lambda_1+\tau\lambda_2)/2\Big),$$

$Y = \tau^{1/\tau}\,\beta_{1,0}^{-1}(\tau, b_0 = \tau)$, *then the distribution*

$$M_{(\tau,\lambda_1,\lambda_2)} = 2\pi\, 2^{-\big[3(1+\tau)+2\tau(\lambda_1+\lambda_2)\big]/\tau}\,\Gamma\big(1-1/\tau\big)^{-1} L\, X_1\, X_2\, X_3\, Y,$$

$$\mathbf{E}\big[M^n_{(\tau,\lambda_1,\lambda_2)}\big] = \mathbf{E}\Big[\Big(\int_0^1 s^{\lambda_1}(1-s)^{\lambda_2}\, M_\beta(ds)\Big)^n\Big],\ n < 1/\beta^2$$

# Multiplicative Chaos on the Interval and Barnes Beta II

**Theorem (CMP '09, IMRN '12)**

$\log M_{(\tau,\lambda_1,\lambda_2)}$ *is infinitely divisible and has a continuous density.*

$$\mathbf{E}\big[M^{-n}_{(\tau,\lambda_1,\lambda_2)}\big] = \prod_{k=0}^{n-1} \frac{\Gamma\big(2+\lambda_1+\lambda_2+\frac{(n+2+k)}{\tau}\big)\Gamma\big(1-\frac{1}{\tau}\big)}{\Gamma\big(1+\lambda_1+\frac{(k+1)}{\tau}\big)\Gamma\big(1+\lambda_2+\frac{(k+1)}{\tau}\big)\Gamma\big(1+\frac{k}{\tau}\big)}$$

*If* $\widetilde{M}_{(\tau,\lambda_1,\lambda_2)} \overset{\text{in law}}{=} LN,\; L = \exp\big(\mathcal{N}(0,\, 4\log 2/\tau)\big)$, *and some* $N > 0$,

$$\mathbf{E}\big[\widetilde{M}^{-n}_{(\tau,\lambda_1,\lambda_2)}\big] = \mathbf{E}\big[M^{-n}_{(\tau,\lambda_1,\lambda_2)}\big] \rightarrow \widetilde{M}_{(\tau,\lambda_1,\lambda_2)} \overset{\text{in law}}{=} M_{(\tau,\lambda_1,\lambda_2)}$$

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**Conjecture (CMP '09, Fyodorov *et al* '09, IMRN '12)**

$$M_{(\tau,\lambda_1,\lambda_2)} \overset{\text{in law}}{=} \int_0^1 s^{\lambda_1}(1-s)^{\lambda_2}\, M_\beta(ds),\;\; \tau = 1/\beta^2 > 1$$

# GFF and Multiplicative Chaos on the Unit Circle

- GFF on the unit circle: $V_\varepsilon(\theta)$, $\theta \in [-\pi, \pi)$, centered gaussian,

$$\mathbf{Cov}\left[V_\varepsilon(\theta),\ V_\varepsilon(\xi)\right] = \begin{cases} -2\,\log\left|e^{i\psi} - e^{i\xi}\right|, \ |\xi - \psi| \gg \varepsilon, \\ 2\left(\kappa - \log\varepsilon\right), \psi = \xi,\ \kappa \geq 0. \end{cases}$$

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- Kahane-Bacry-Muzy: $e^{-\beta^2(\kappa - \log\varepsilon)} \int_a^b e^{\beta V_\varepsilon(\theta)}\, d\theta \to M_\beta(a, b)$ as $\varepsilon \to 0$, and $\mathbf{E}[M_\beta(a, b)] = |b - a|$ if $0 \leq \beta < 1$

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- Moments are given by the Morris integral: let $n < 1/\beta^2$,

$$\mathbf{E}\Big[\Big(\int_{-\pi}^{\pi} e^{i\theta(\lambda_1-\lambda_2)/2}\,|1+e^{i\theta}|^{\lambda_1+\lambda_2}\, M_\beta(d\theta)\Big)^n\Big] =$$

$$\int\limits_{[-\pi,\pi]^n} \prod_{l=1}^{n} e^{i\theta_l(\lambda_1-\lambda_2)/2}|1+e^{i\theta_l}|^{\lambda_1+\lambda_2} \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^{-2\beta^2}\, d\theta_1\cdots d\theta_n$$

# Multiplicative Chaos on the Circle and Barnes Beta I

**Theorem (Fyodorov and Bouchaud '08 ($\lambda_1 = \lambda_2 = 0$), DO '16)**

*Let* $\tau = 1/\beta^2 > 1$ *and define*

$$X = \beta_{2,2}^{-1}(\tau, b_0 = \tau, \; b_1 = 1 + \tau\lambda_1, \; b_2 = 1 + \tau\lambda_2),$$
$$Y = \tau^{1/\tau}\beta_{1,0}^{-1}(\tau, b_0 = \tau(\lambda_1 + \lambda_2 + 1) + 1),$$

*then* $M_{(\tau,\lambda_1,\lambda_2)} = \frac{2\pi}{\Gamma(1-1/\tau)} XY$ *has the property that for* $n < 1/\beta^2$

$$\mathbf{E}\Big[M^n_{(\tau,\lambda_1,\lambda_2)}\Big] = \mathbf{E}\Big[\Big(\int_{-\pi}^{\pi} e^{i\theta(\lambda_1-\lambda_2)/2}\, |1 + e^{i\theta}|^{\lambda_1+\lambda_2}\, M_\beta(d\theta)\Big)^n\Big].$$

$\log M_{(\tau,\lambda_1,\lambda_2)}$ *is infinitely divisible. If* $\lambda_1 = \lambda_2 = 0$,

$$M_{(\tau,0,0)} = \frac{2\pi\tau^{1/\tau}}{\Gamma(1-1/\tau)}\beta_{1,0}^{-1}(\tau, b_0 = \tau)$$

# Multiplicative Chaos on the Circle and Barnes Beta II

**Theorem (DO '16)**

$$\mathbf{E}[M^{-n}_{(\tau,\lambda_1,\lambda_2)}] = \frac{1}{(2\pi)^n}\prod_{j=0}^{n-1}\frac{\Gamma(1+\lambda_1+\frac{(1+j)}{\tau})\Gamma(1+\lambda_2+\frac{(1+j)}{\tau})\Gamma(1-\frac{1}{\tau})}{\Gamma(1+\lambda_1+\lambda_2+\frac{(1+j)}{\tau})\Gamma(1+\frac{j}{\tau})}$$

*The Stieltjes moment problem for* $M^{-1}_{(\tau,\lambda_1,\lambda_2)}$ *is determinate*

# Multiplicative Chaos on the Circle and Barnes Beta II

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$$\mathbf{E}[M_{(\tau,\lambda_1,\lambda_2)}^{-n}] = \frac{1}{(2\pi)^n} \prod_{j=0}^{n-1} \frac{\Gamma(1+\lambda_1+\frac{(1+j)}{\tau})\Gamma(1+\lambda_2+\frac{(1+j)}{\tau})\Gamma(1-\frac{1}{\tau})}{\Gamma(1+\lambda_1+\lambda_2+\frac{(1+j)}{\tau})\Gamma(1+\frac{j}{\tau})}$$

*The Stieltjes moment problem for $M_{(\tau,\lambda_1,\lambda_2)}^{-1}$ is determinate*

**Conjecture (Fyodorov and Bouchaud '08 ($\lambda_1 = \lambda_2 = 0$), DO '16)**

$$M_{(\tau,\lambda_1,\lambda_2)} \overset{\text{in law}}{=} \int_{-\pi}^{\pi} e^{i\theta(\lambda_1-\lambda_2)/2} \, |1+e^{i\theta}|^{\lambda_1+\lambda_2} \, M_\beta(d\theta), \; \tau = 1/\beta^2 > 1$$

# Self-Duality of the Mellin Transform of Chaos Measures

- Let $\mathfrak{M}(q \,|\, \tau, \lambda_1, \lambda_2) = \mathbf{E}\big[M^q_{(\tau,\lambda_1,\lambda_2)}\big]$ (interval or circle).
  Fyodorov *et al* '09 ($\lambda_1 = \lambda_2 = 0$), '15 (arbitrary $\lambda$ on interval)

$$F(q \,|\, \beta, \lambda_1, \lambda_2) = \mathfrak{M}\big(\frac{q}{\beta} \,\big|\, \frac{1}{\beta^2}, \beta\lambda_1, \beta\lambda_2\big)(2\pi)^{-\frac{q}{\beta}}\Gamma^{\frac{q}{\beta}}(1-\beta^2)\Gamma(1-\frac{q}{\beta}),$$

$$F(q \,|\, \beta, \lambda_1, \lambda_2) = F(q \,\big|\, \frac{1}{\beta}, \lambda_1, \lambda_2) \text{ (self-dual)}$$

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$$F(q\,|\,\beta,\lambda_1,\lambda_2) = \mathfrak{M}\big(\frac{q}{\beta}\,\big|\,\frac{1}{\beta^2},\beta\lambda_1,\beta\lambda_2\big)(2\pi)^{-\frac{q}{\beta}}\Gamma^{\frac{q}{\beta}}(1-\beta^2)\Gamma(1-\frac{q}{\beta}),$$

$$F(q\,|\,\beta,\lambda_1,\lambda_2) = F(q\,\Big|\,\frac{1}{\beta},\lambda_1,\lambda_2)\ \text{(self-dual)}$$

- Forum Math '14 (arbitrary $\lambda$ on interval), DO '16 (circle) involution invariance under $\tau\rightarrow\frac{1}{\tau},\ q\rightarrow\frac{q}{\tau},\ \lambda_i\rightarrow\tau\lambda_i$:

$$\mathfrak{M}(q\,|\,\tau,\lambda_1,\lambda_2)(2\pi)^{-q}\,\Gamma^q(1-\frac{1}{\tau})\Gamma(1-q) =$$

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# Self-Duality of the Mellin Transform of Chaos Measures

- Let $\mathfrak{M}(q\,|\,\tau,\lambda_1,\lambda_2) = \mathbf{E}\left[M^q_{(\tau,\lambda_1,\lambda_2)}\right]$ (interval or circle). Fyodorov *et al* '09 ($\lambda_1=\lambda_2=0$), '15 (arbitrary $\lambda$ on interval)

$$F(q\,|\,\beta,\lambda_1,\lambda_2) = \mathfrak{M}\Big(\frac{q}{\beta}\,\Big|\,\frac{1}{\beta^2},\beta\lambda_1,\beta\lambda_2\Big)(2\pi)^{-\frac{q}{\beta}}\Gamma^{\frac{q}{\beta}}(1-\beta^2)\Gamma(1-\frac{q}{\beta}),$$

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$$M_{(\tau,\lambda_1,\lambda_2)} = 2^{-[3(1+\tau)+2\tau(\lambda_1+\lambda_2)]/\tau}L\,X_1\,X_2\,X_3\boxed{2\pi\,\Gamma\big(1-1/\tau\big)^{-1}Y}$$

# Maximum of GFF and the Freezing Hypothesis

- Problem formulation: let $V_\varepsilon(x)$ be GFF on $[0, 1]$, $N = 1/\varepsilon$, $x_j = j\varepsilon$, $j = 1 \dots N$, $\lambda_1$, $\lambda_2 \geq 0$. In the limit $N \to \infty$, what is

$$V_N \equiv \max\{V_\varepsilon(x_j) + \lambda_1 \log(x_j) + \lambda_2 \log(1 - x_j),\ j = 1 \cdots N\}?$$

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- Fyodorov *et al* '09, '12, '15: for $0 < \beta < 1$,
$$Z_{\lambda_1,\lambda_2,N}(\beta) = \sum_{j=1}^{N} x_j^{\beta\lambda_1}(1 - x_j)^{\beta\lambda_2} e^{\beta V_\varepsilon(x_j)} \approx N^{1+\beta^2} e^{\beta^2\kappa} M_{(\tau,\beta\lambda_1,\beta\lambda_2)},$$
$$\mathbf{E}[e^{qV_N}] \approx e^{q(2\log N - (3/2)\log\log N + O(1))} F(q \,|\, \beta = 1, \lambda_1, \lambda_2)$$
$$F(q \,|\, \beta = 1, \lambda_1, \lambda_2) = \mathbf{E}\left[\lim_{\beta\uparrow 1} M_{(\frac{1}{\beta^2},\lambda_1,\lambda_2)} \frac{\Gamma(1-\beta^2)}{2\pi}\right]^q \Gamma(1 - q)$$

# Maximum of GFF on the Interval and Critical Selberg Law

**Conjecture (Fyodorov *et al* '09, '12, '15)**

$$V_N \approx 2\log N - \frac{3}{2}\log\log N + C + \log \lim_{\beta\uparrow 1}\Big[M_{(\frac{1}{\beta^2},\lambda_1,\lambda_2)}\frac{\Gamma(1-\beta^2)}{2\pi}\Big] + \log Y'$$

**Theorem (Forum Math '14, DO '16)**

$$\lim_{\beta\uparrow 1}\Big[M_{(\frac{1}{\beta^2},\lambda_1,\lambda_2)}\frac{\Gamma(1-\beta^2)}{2\pi}\Big] = \log(L) + \sum_{j=1}^{3}\log X_j + \log Y,$$

$$\log(L) = \mathcal{N}(0,\,4\log 2),\ \ Y = Y' = \beta_{1,0}^{-1}(1, b_0 = 1),$$

$$X_1 = \beta_{2,2}^{-1}(1, b_0 = 2+\lambda_1,\ b_1 = (\lambda_2-\lambda_1)/2,\ b_2 = (\lambda_2-\lambda_1)/2),$$

$$X_2 = \beta_{2,2}^{-1}(1, b_0 = 2+(\lambda_1+\lambda_2)/2,\ b_1 = 1/2,\ b_2 = 1/2),$$

$$X_3 = \beta_{2,2}^{-1}(1, b_0 = 2,\ b_1 = (2+\lambda_1+\lambda_2)/2,\ b_2 = (2+\lambda_1+\lambda_2)/2)$$

# Maximum of GFF on the Circle and Critical Morris Law

**Conjecture (Fyodorov *et al* '09, '12, '15, DO '16)**

$$V_N \approx 2\log N - \frac{3}{2}\log\log N + C + \log \lim_{\beta\uparrow 1}\Big[M_{(\frac{1}{\beta^2},\lambda_1,\lambda_2)}\frac{\Gamma(1-\beta^2)}{2\pi}\Big] + \log Y'$$

**Theorem (Fyodorov and Bouchaud '08 ($\alpha = 0$), DO '16)**

$$\lim_{\beta\uparrow 1}\Big[M_{(\frac{1}{\beta^2},\lambda_1,\lambda_2)}\frac{\Gamma(1-\beta^2)}{2\pi}\Big] = \log X + \log Y,$$

$$X = \beta_{2,2}^{-1}(1, b_0 = 1, b_1 = 1+\alpha, b_2 = 1+\alpha),$$

$$Y = \beta_{1,0}^{-1}(1, b_0 = 2\alpha + 2),$$

$$Y' = \beta_{1,0}^{-1}(1, b_0 = 1).$$

*In particular, if* $\alpha = 0$, $\log X + \log Y = \log Y'$

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- Similarly, $\mathcal{H}^{1/2}(\mathbb{T})$ statistics (eg. global CUE) give rise in the same way to Morris integral distribution

# Open Questions and Work in Progress

- Fundamental open problem: total mass of chaos measures. Can we compute negative moments from first principles?
- What is the *right* approximation of the GFF? GFF on the interval is the limit of a fractional integral of Brownian motion. Is the law of the exponential functional of BM à la Marc Yor (Hartman-Watson law) related to the Selberg integral distribution? GFF and hyperbolic geometry?
- Why do Barnes beta distributions appear in the distribution of extremes (stable Lévy à la Kuznetsov, GFF conjecturally)?
- What is the origin of infinite divisibility?