Extreme values of the GUE characteristic polynomial

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Workshop on extrema of logarithmically correlated processes.
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$$p_N(x) = \det(xI - H)$$

- In particular, we are interested in **extreme values** of $p_N(x)$ as x varies in $[-1, 1]$.

Statistics of the maximum

What can we say about the statistics of the **highest peak** of the characteristic polynomial?

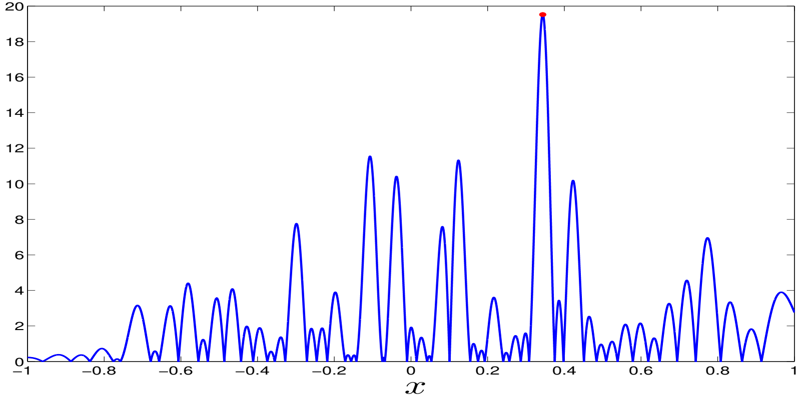

Figure : A single realization of $|p_N(x)|e^{-\mathbb{E}\log|p_N(x)|}$. What are the statistics of the **maximum value**? (see the red dot)

Larger N: logarithmic scale

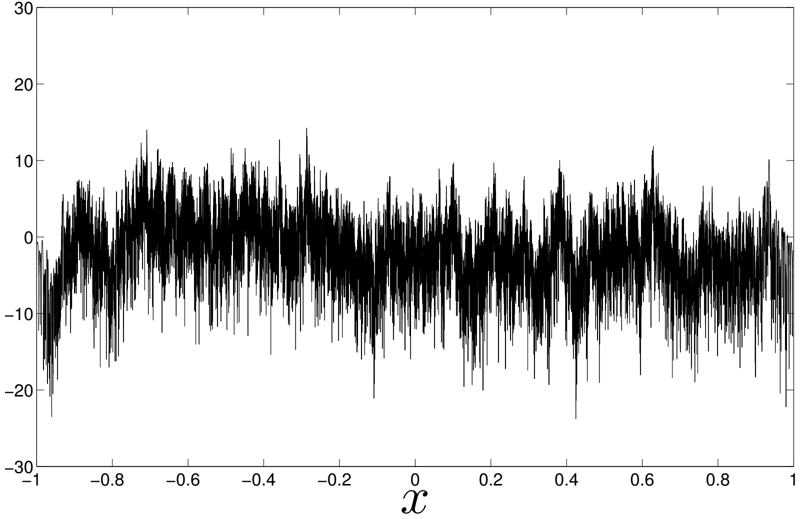

Figure : A plot of a single realization of $\log |p_N(x)| - \mathbb{E} \log |p_N(x)|$ for $N = 3000$.

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Conjecture (Fyodorov and Simm '15)

We have the convergence in distribution

$$M_N^* - 2\log(N) + \frac{3}{2}\log(\log(N)) \overset{d}{\Longrightarrow} u, \qquad N \to \infty,$$

where u is a continuous random variable characterized by

$$\mathbb{E}(e^{-us}) = \frac{1}{C}K^s \frac{\Gamma(s+1)\Gamma(s+3)G(s+7/2)^2}{G(s+1)G(s+6)}$$

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$2\log(N)$ term recently obtained by Lambert and Paquette '16 (unpublished).

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Our Laplace transform formula explicitly characterises the so-called 'derivative martingale' in this example. We can write

$$P(y) = \lim_{N\to\infty} \mathbb{P}(M_N^* \geq m_n - y) = \mathbb{E}(e^{-e^{y-z}})$$

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We have the decomposition as independent r.v.'s:

$$z = \log(K) + Y + \log \beta_{2,2}$$

where $Y \in \mathbb{R}$ has density $p_Y(y) = \frac{1}{2}e^{3y}e^{-e^y}$ and $\beta_{2,2} \in (0,1)$ is a particular *Barnes Beta random variable*, studied in works of Ostrovsky 2009-2016 (*see later today!*).

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2. Compute moments of the partition function $\mathcal{Z}_N(\beta)$ and asymptotics of Hankel determinants.
3. Analytic continuation of moments into the complex plane.
4. Duality $\beta \to 1/\beta$, freezing and reconstruction of M_N^* in the low-temperature phase.

Analytical approach

Random partition function:

$$\boxed{\mathcal{Z}_N(\beta) = \frac{N}{2}\int_{-1}^{1} e^{\beta V_N(x)}\rho(x)^q\,dx}$$

with potential $V_N(x) = 2(\log|p_N(x)| - \mathbb{E}\log|p_N(x)|)$.

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Factors $N/2$, $\rho(x) = (2/\pi)\sqrt{1 - x^2}$ and $q > 0$, etc. Why? Needed later.

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Goal: Compute the right-hand side in (1).

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The expectation is a *Hankel determinant*:

$$\mathbb{E}\left(\prod_{j=1}^{k} |\det(x_j - H)|^{2\beta}\right) \propto \det\left\{\int_{\mathbb{R}} \lambda^{i+j-2} w_{x,k}^{(\beta)}(\lambda)\, d\lambda\right\}_{i,j=1}^{N}$$

where

$$w_{x,k}^{(\beta)}(\lambda) = e^{-2N\lambda^2} \prod_{j=1}^{k} |x_j - \lambda|^{2\beta}$$

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$$\mathbb{E}\left(e^{-\beta\sum_{j=1}^{k}V_N(x_j)}\right)=N^{(1+\beta^2)k}\prod_{j=1}^{k}C(\beta)(1-x_j^2)^{\beta^2/2}\prod_{1\leq i<j\leq k}|x_i-x_j|^{-2\beta^2}[1+o(1)]$$

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where $C(\beta) = 2^{-\beta^2} G(\beta+1)^2 / G(2\beta+1)$. (Brutally) inserting these asymptotics into the moments gives:

$$\mathbb{E}(\mathcal{Z}_N(\beta)^k) \sim c_\beta^k N^{(1+\beta^2)k} \int_{[-1,1]^k} \prod_{j=1}^{k} dx_j \,(1-x_j^2)^{a/2} \prod_{1\le i<j\le k} |x_i - x_j|^{-2\beta^2}$$

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Problem: uniformity in $x_1,\ldots,x_k$. *Case $k=2$: Claeys and Krasovsky '14, Claeys and Fahs '15.* Conjecture $k \ge 2$:

$$\mathbb{E}(\mathcal{Z}_N(\beta)^k) = \begin{cases} O(N^{(1+\beta^2)k}), & \beta^2 < 1/k \\ O(N^{1+k^2\beta^2}), & \beta^2 > 1/k \end{cases}$$

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which implies that $\mathbb{E}((\mathcal{Z}_N(\beta)/c_\beta)^{-s/\beta}) = \Gamma(1+s\beta)m_\beta(s)$ where

$$m_\beta(s) \propto \frac{G_\beta(2s+2\overline{a}_\beta+\frac{2}{\beta}+2\beta)}{G_\beta(s+\beta+\frac{1}{\beta})G_\beta(s+\overline{a}_\beta+\frac{1}{\beta}+\beta)G_\beta(s+\overline{a}_\beta+\frac{1}{\beta}+\beta)G_\beta(s+2\overline{a}_\beta+\frac{2}{\beta}+2\beta)}$$

where $\overline{a}_\beta = q/\beta+\beta$, for any $-s/\beta = 0, 1, 2, 3, \ldots$ and defines a particular continuation to the entire complex plane [see also Fyodorov and Le Doussal '15]

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Rigorously proving these conjectures represents an ongoing and significant mathematical challenge.

Numerical evidence

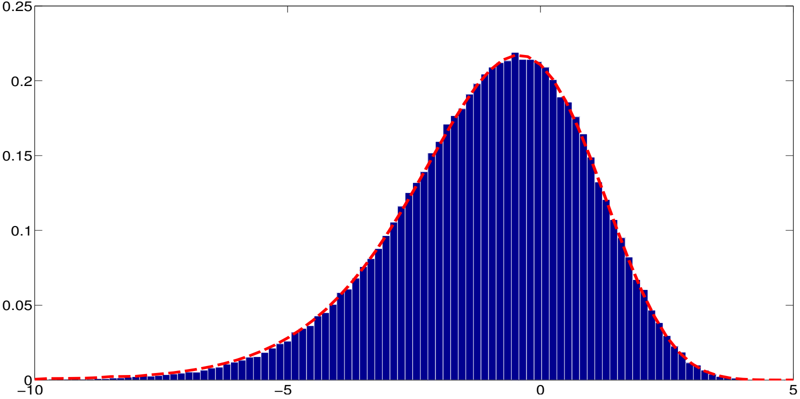

Figure : Histogram of 250,000 random samples of M_N^* with $N = 3000$. Red curve: probability density $p(x)$ of u obtained from inverse Laplace transform.

The 'Carpentier Le Doussal tail' $p(x) \sim -xe^x$ as $x \to -\infty$ expected to be universal for log-correlated fields.

Thank you!

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Advert: Series of talks (reading seminar) on logarithmically correlated processes at Warwick University, Tuesdays 1pm. Everyone welcome to attend / give talks!