RH in Characteristic p: the importance of family values

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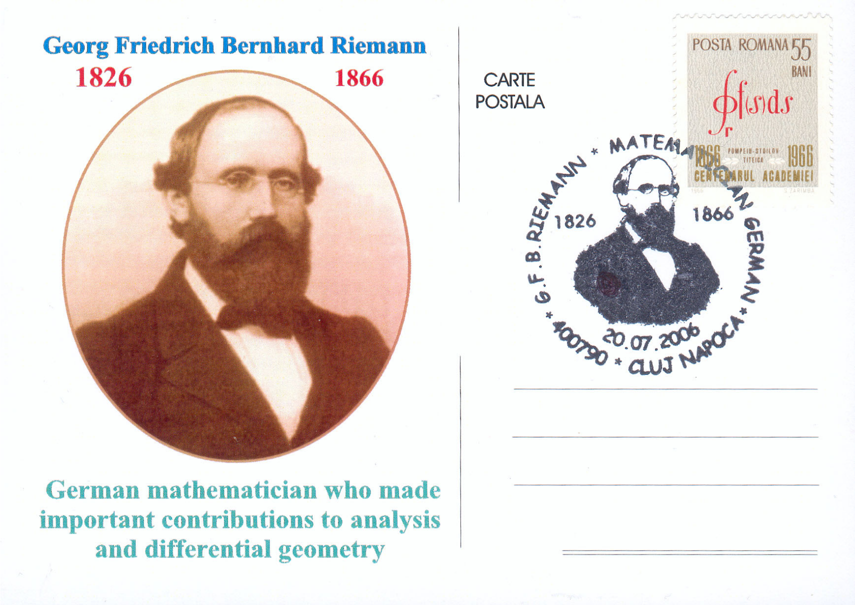
1859: publication of
Über die Anzahl der Primzahlen unter einer gegebenen Grösse, 10 pages!

Riemann studies $\zeta(s)$, formulates Riemann Hypothesis on fourth page, saying "es ist sehr wahrscheinlich..." ("it is very likely that...").


Georg Friedrich Bernhard Riemann
1826
1866
German mathematician who made important contributions to analysis and differential geometry
CARTE POSTALA
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1866 1966
CENTENARUL ACADEMIEI
G.F.B. RIEMANN
GERMAN
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1920-1949
see Roquette’s “The Riemann hypothesis in characteristic p, its origin and development"

(function fields of) curves over $\mathbb{F}_q$ as analogues of number fields.

Riemann wanted to count primes (“closed points") of norm up to T;

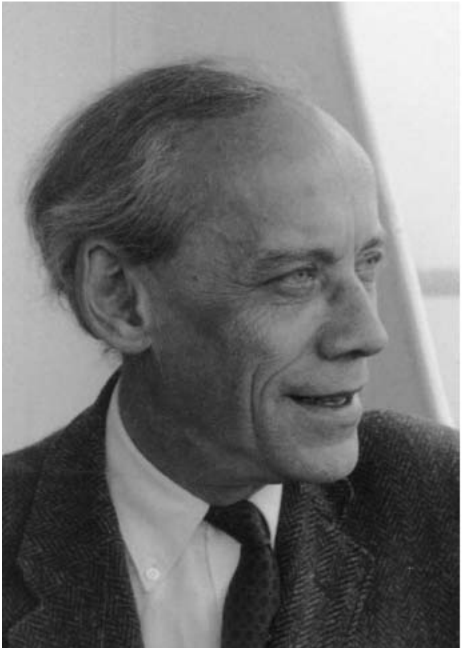
for curve $C/\mathbb{F}_q$, this amounts to counting $C(\mathbb{F}_{q^n})$; a closed point $\mathcal{P}$ of norm $\mathbb{N}\mathcal{P} = q^n$ contributes n points to this set.

Notice here that $n = \log_q(\mathbb{N}\mathcal{P})$, analogous to counting p with multiplicity $\log(p)$.

1920 thesis (published 1923) Artin; defines Zeta for curves, formulates RH

1931: F.K. Schmidt; correct shape for Zeta of curves

Artin



F.K. Schmidt

counting points
1930;Davenport; $y^2 = (x + a_1)...(x + a_k)$ over $\mathbb{F}_p$
1930; Mordell; $y^n = x^m$ + lower, over $\mathbb{F}_p$

what they wanted: control of the error term as p varies

ideally $O(p^{1/2})$; they got results like $O(p^{3/4})$ or $O(p^{2/3})$

Hasse teased them: “Have you reduced any exponents lately?"

Their response: “So use your fancy ‘intrinsic’ point of view to do better!"

Davenport

Mordell

a key insight, due to Artin, is that one should **not** vary p, but rather vary the extensions $\mathbb{F}_q$ of a given $\mathbb{F}_p$, and think of counting $\mathbb{F}_q$ points; then one sees that

RH **is** control of error term as q varies in the fixed characteristic p.

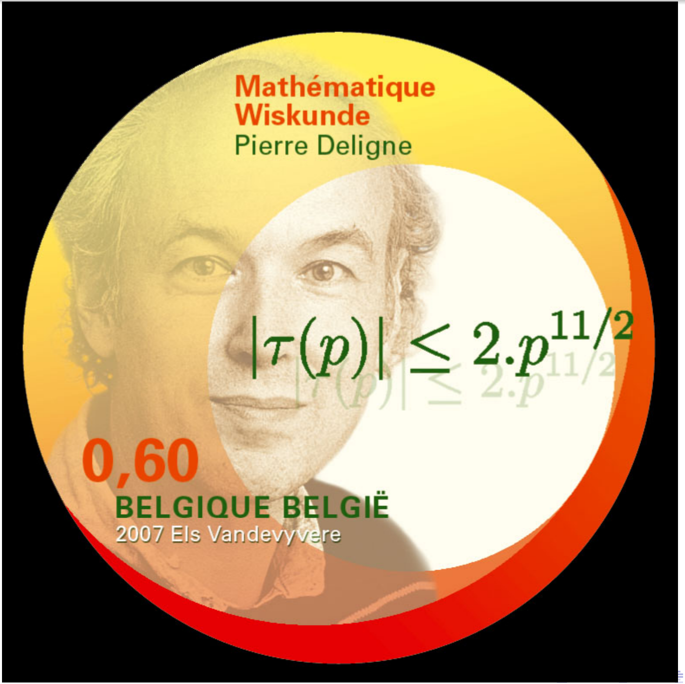
1933,1936; Hasse; proves RH in genus one

Hasse (standing) with Artin

1948; Weil; proves RH for curves of arb. genus.
1949: Weil formulates RH for (projective, smooth) varieties of any dimension (“Weil Conjectures")

Weil

1973: Deligne proves Weil Conjectures, makes essential use of families, as incarnated in the “local systems" of Grothendieck and their (compact) cohomology


Mathématique
Wiskunde
Pierre Deligne
$|\tau(p)| \leq 2.p^{11/2}$
0,60
BELGIQUE BELGIË
2007 Els Vandevyvere

a baby version of Deligne's proof, in the case of curves, using families