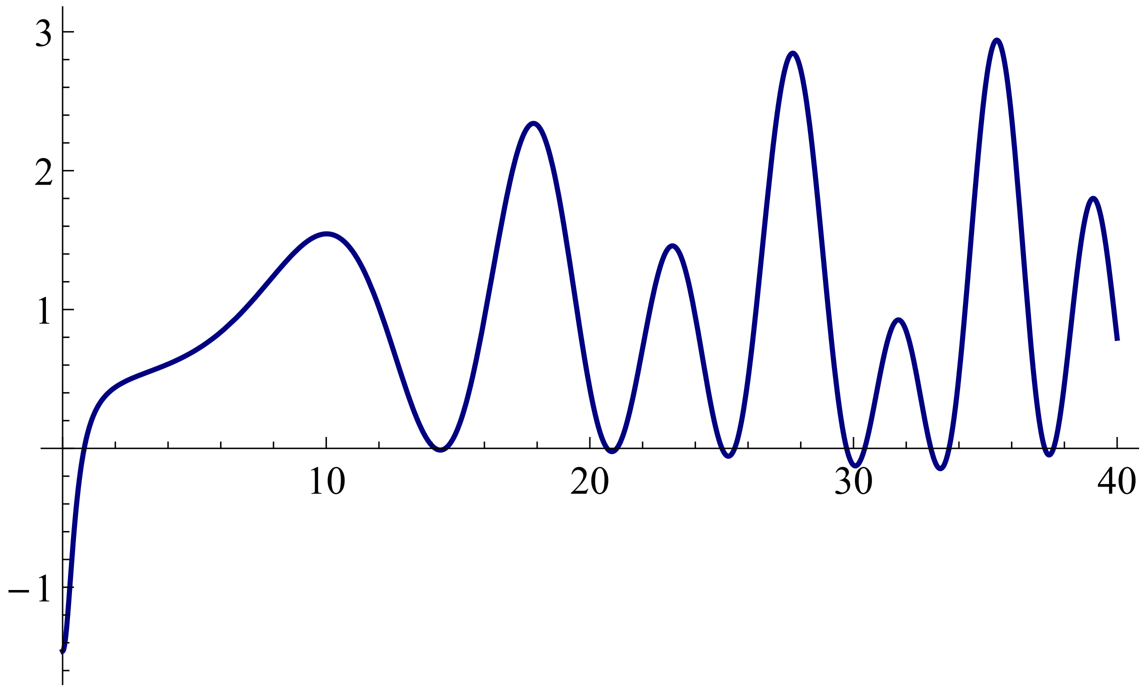
Rational approximations to ζ

Keith Ball



This talk describes rational functions approximating ζ:

$$\frac{1}{(s-1)}, \quad \frac{s+1}{2(s-1)}, \quad \frac{4s^2+11s+9}{6(s+3)(s-1)}, \quad \frac{(s+2)(3s^2+10s+11)}{4(s^2+6s+11)(s-1)},$$

$$\frac{(s+2)(72s^3+490s^2+1193s+1125)}{30(3s^3+29s^2+106s+150)(s-1)}, \ldots$$

The small size of $\zeta(1/2+it)$ depends upon cancellation between different Dirichlet terms.

Each coefficient in the rational functions depends upon all the Dirichlet terms so the cancellation is built into the coefficients.

For each integer $m \geq 0$ we define

$$p_m(t) = (1-t)\left(1-\frac{t}{2}\right)\dots\left(1-\frac{t}{m}\right)$$

and the coefficients $(a_{m,j})$ by

$$p_m(t) = \sum_0^m (-1)^j a_{m,j} t^j.$$

We then set

$$F_m(s) = \sum_0^m \frac{a_{m,j} B_j}{s+j-1}$$

and

$$G_m(s) = \sum_{j=0}^m (-1)^j \frac{a_{m,j}}{s+j-1}.$$

The rational functions in question are the ratios

$$\frac{F_m(s)}{(s-1)G_m(s)}.$$

For example

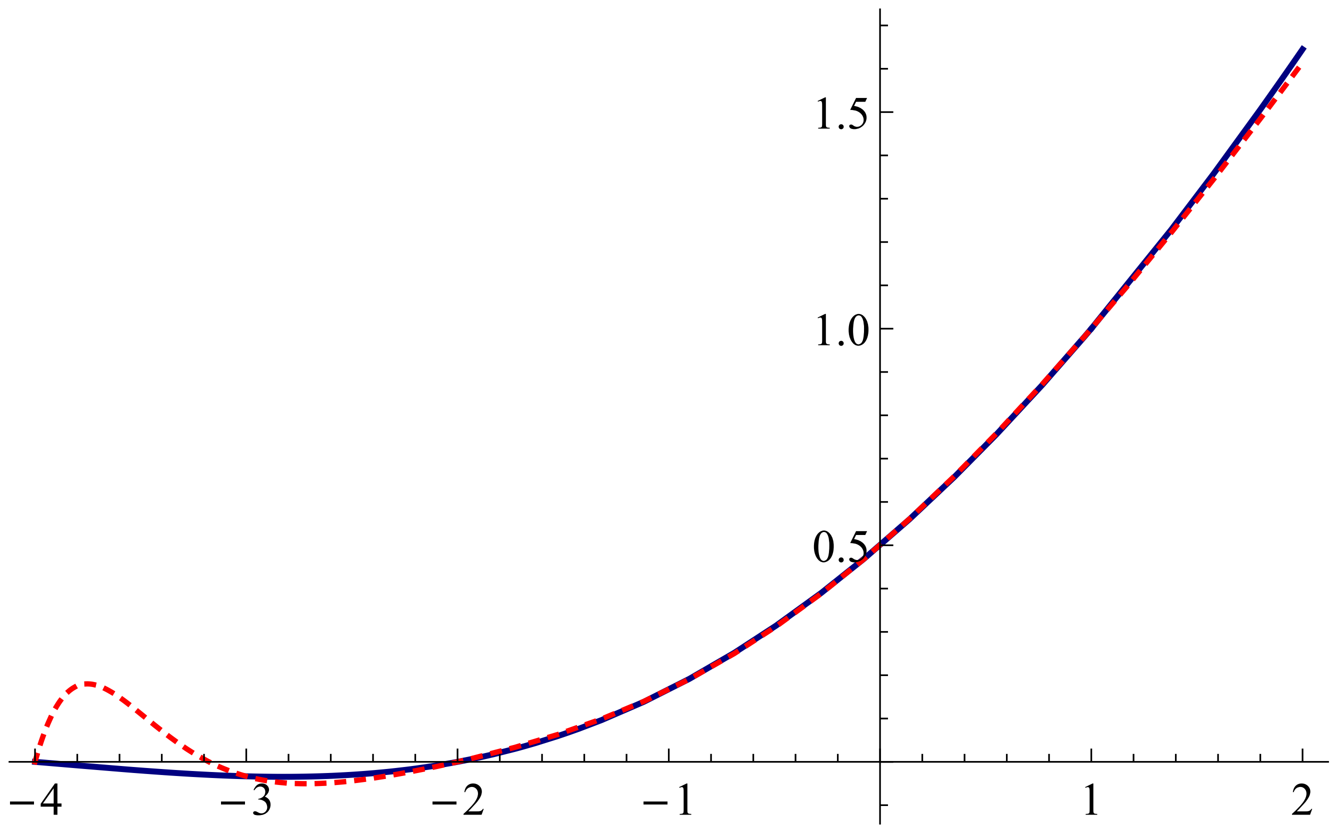
$$F_3(s) = \frac{1}{s-1} - \frac{11}{12s} + \frac{1}{6(s+1)} = \frac{3s^2+10s+11}{12(s-1)s(s+1)}$$

and

$$G_3(s) = \frac{1}{s-1} - \frac{11}{6s} + \frac{1}{s+1} - \frac{1}{6(s+2)} = \frac{s^2+6s+11}{3(s-1)s(s+1)(s+2)}.$$

The m^{th} ratio interpolates ζ at the points $0, -1, -2, \ldots, 1-m$ and has a simple pole with residue 1 at $s = 1$.

The graph shows $(s-1)\zeta(s)$ and the ratio $F_5(s)/G_5(s)$



The sequence converges locally uniformly to ζ, at least to the right of the line $\{s : \Re s = 0\}$.

We shall see that

$$F_m(s) \approx h_m^{1-s}\Gamma(s)\zeta(s)$$

and

$$(s-1)G_m(s) \approx h_m^{1-s}\Gamma(s)$$

where h_m is the partial sum $\sum_{j=1}^{m} 1/j$ of the harmonic series.

The rational functions might still be difficult to analyse: what are the coefficients?

Focus on the F_m:

$$\begin{array}{cccc} F_0(s), & F_1(s), & F_2(s), & F_3(s) \\[2ex] \dfrac{1}{(s-1)}, & \dfrac{s+1}{2(s-1)s}, & \dfrac{4s^2+11s+9}{12(s-1)s(s+1)}, & \dfrac{(s+2)(3s^2+10s+11)}{12(s-1)s(s+1)(s+2)} \end{array}$$

We have a recurrence relation: for each m

$$(s+m-1)\,F_m(s) = \frac{1}{(m+1)} + (m+1)\sum_{j=1}^{m} \frac{F_{m-j}(s)}{j(j+1)}.$$

Equivalently

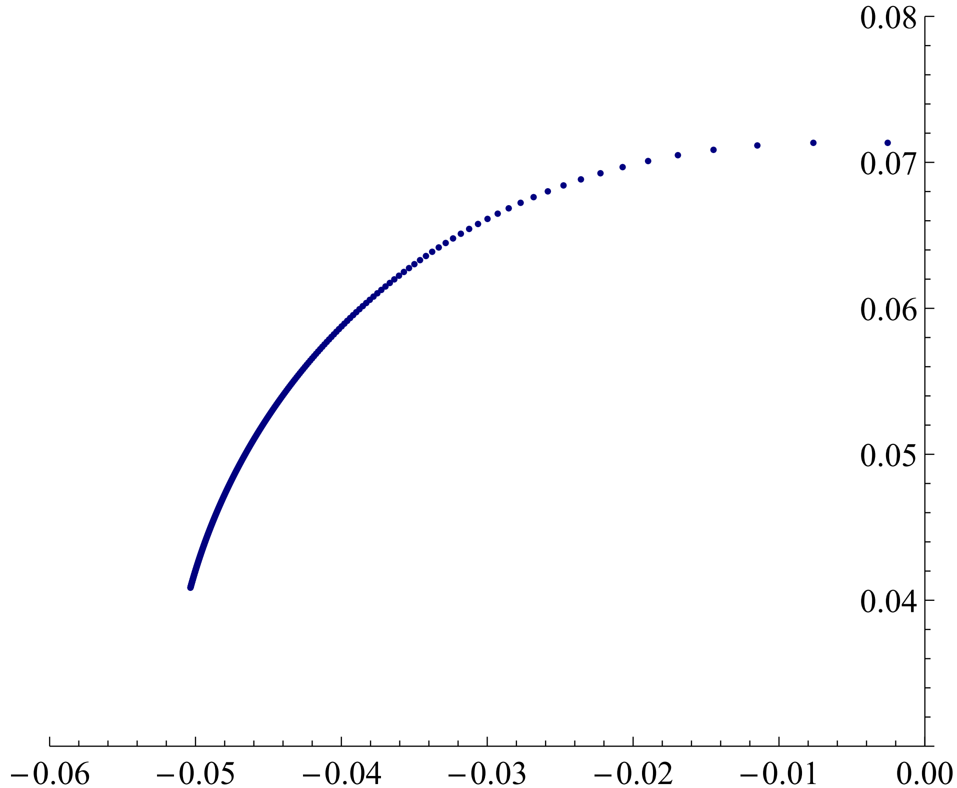
$$\left(1+\frac{s-1}{m}\right)F_m(s)=\frac{1}{m(m+1)}+\frac{m+1}{m}\sum_{j=1}^{m}\frac{F_{m-j}(s)}{j(j+1)}.$$

At each stage we take a weighted average of the previous terms, add a small bit and rotate slightly.

This is a very stable dynamical system.

The dependence of the end result ζ on s can be very sensitive because s rotates at each step. But for each fixed s we have a very smooth way of getting to $\zeta(s)$.

Here are the first few hundred values of $(n+1)F_n(1/2-14i)$.



If we treat the first $m+1$ of these relations as a linear system for the values $F_0(s), F_1(s), \ldots, F_m(s)$ we can express the fact that $F_m(s) = 0$ by the vanishing of a certain determinant.

The numerator of the m^{th} function is the determinant of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m} & \frac{1}{m-1} & \ldots & \frac{1}{2} & 1 \end{pmatrix} + (1-s) \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & 0 & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \frac{1}{m} \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

So RH can be restated as what looks like a rather conventional spectral problem.

Connes reformulated RH as a statement about the spectrum of an operator acting on an infinite-dimensional function space.

There is a connection between Connes' infinite-dimensional operator and these finite-dimensional ones.

If $\Re s > 1$

$$\begin{aligned} G_m(s) &= \sum_{j=0}^{m} (-1)^j \frac{a_{m,j}}{s+j-1} = \sum_{j=0}^{m} (-1)^j a_{m,j} \int_0^1 x^j x^{s-2}\, dx \\ &= \int_0^1 p_m(x) x^{s-2}\, dx \end{aligned}$$

$$p_m(x) = (1-x)\left(1-\frac{x}{2}\right)\ldots\left(1-\frac{x}{m}\right) \approx e^{-h_m x}$$

so it is no surprise that $G_m(s) \approx h_m^{1-s}\Gamma(s-1)$.

We want to do something similar for F_m.

If $\Re s > 1$

$$\int_0^\infty \frac{y}{1-e^{-y}} e^{-y} y^{s-2}\, dy = \int_0^\infty \left(\sum_{n=1}^\infty e^{-ny} \right) y^{s-1}\, dy$$

$$= \sum_{n=1}^\infty \int_0^\infty e^{-ny} y^{s-1}\, dy = \sum_{n=1}^\infty \frac{1}{n^s} \Gamma(s).$$

So

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{-\log(1-(1-e^{-y}))}{1-e^{-y}} e^{-y} y^{s-2}\, dy$$

$$= \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1} (1-e^{-y})^k\, e^{-y} y^{s-2}\, dy.$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1}(1-e^{-y})^k\, e^{-y} y^{s-2}\, dy.$$

Using a standard formula for Bernoulli numbers we get that for $\Re s > 1$

$$F_m(s) \;=\; \int_0^1 \left(\sum_{k=0}^m \frac{1}{k+1} \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \right) x^{s-2}\, dx$$

If x is close to zero then

$$\begin{aligned}
\Delta_{m,k}(x) \;&=\; \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \\
&\approx\; \sum_{r=0}^k \binom{k}{r} (-1)^r e^{-h_m(r+1)x} = (1-e^{-h_m x})^k\, e^{-h_m x}.
\end{aligned}$$

For small values of x the integrand is approximately

$$\left(\sum_{k=0}^{m} \frac{1}{k+1} e^{-h_m x}(1-e^{-h_m x})^k\right) x^{s-2}.$$

If the approximation were good for *all* x between 0 and 1 then $F_m(s)$ would be close to

$$\int_0^1 \sum_{k=0}^{m} \frac{1}{k+1} e^{-h_m x}(1-e^{-h_m x})^k x^{s-2}\,dx$$

$$= h_m^{1-s} \int_0^{h_m} \sum_{k=0}^{m} \frac{1}{k+1} e^{-y}(1-e^{-y})^k y^{s-2}\,dy$$

and the integral converges to $\Gamma(s)\zeta(s)$ as $m \to \infty$.

We want to show that

$$h_m^{s-1} F_m(s) \to \Gamma(s)\zeta(s)$$

locally uniformly for $\Re s > 0$.

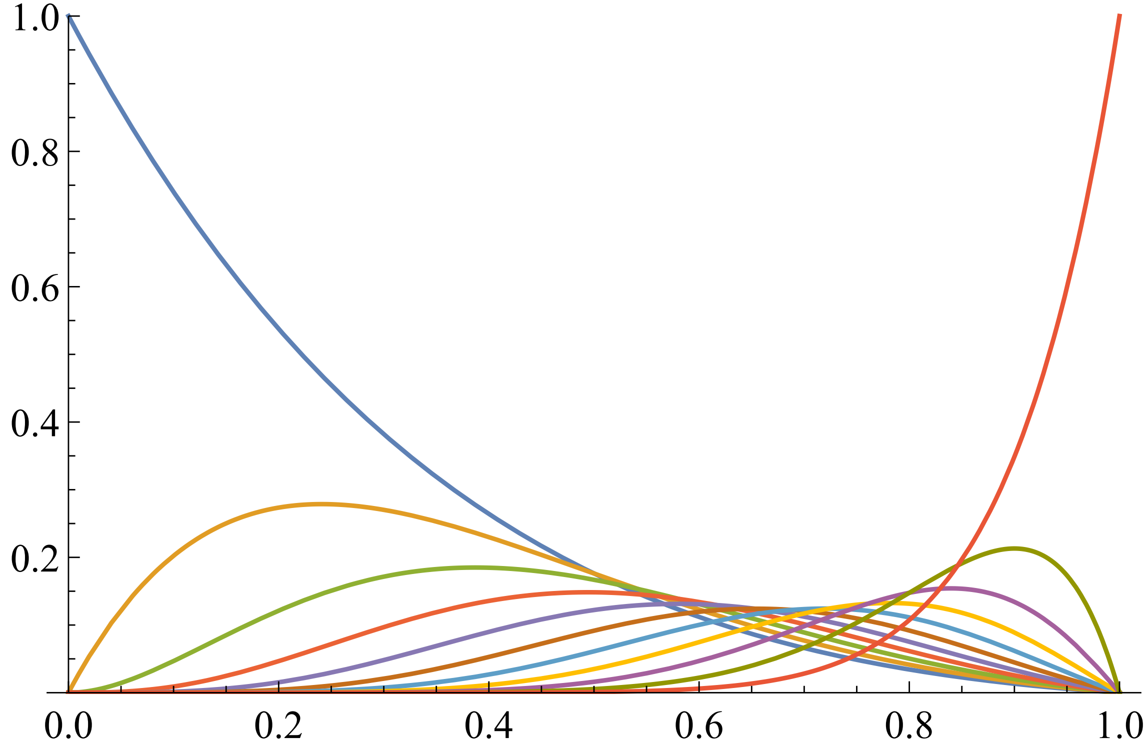
Crossing the pole at $s = 1$ is not the problem.

The difficulty is that unless x is very close to 0, the expressions

$$\Delta_{m,k}(x) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r p_m((r+1)x)$$

involve values of p_m at points well outside the interval $[0, 1]$.

The graph shows the $\Delta_{m,k}$ for $m = 10$.



Lemma 1 (**Key Lemma**)**.** *If m is a non-negative integer, k is any integer and $x \in [0,1]$ then*

$$\Delta_{m,k}(x) \geq 0.$$

It is trivial to check that

$$\sum_{k=0}^{m} \Delta_{m,k}(x) = 1$$

for all x, so the $\Delta_{m,k}$ form a partition of unity on $[0,1]$.

After some fairly delicate estimates we get that the ratios

$$\frac{F_m(s)}{(s-1)G_m(s)}$$

converge locally uniformly to $\zeta(s)$ for $\Re s > 0$.

My guess is that they do so on the entire complex plane.

Theorem 2 (Convergence).

$$h_m^{s-1}(s-1)F_m(s) \to (s-1)\Gamma(s)\zeta(s)$$

locally uniformly for $\Re s > 0$.

Lemma 1 (**Key Lemma**). *If m is a non-negative integer, k is any integer k and $x \in [0,1]$*

$$\Delta_{m,k}(x) \geq 0.$$

The proof of the key lemma involves the introduction of an additional parameter. For each v define

$$P_m(v,x) = (v+1-x)(v+2-x)\dots(v+m-x)$$

and

$$\tilde{\Delta}_{m,k}(v,x) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r P_m(v,(r+1)x).$$

$\tilde{\Delta}_{m,k}(0,x) = m!\Delta_{m,k}(x)$ so the key lemma follows from:

Lemma 3. *If m is a non-negative integer, k is an integer, $v \geq 0$ and $0 \leq x \leq 1$ then*

$$\tilde{\Delta}_{m,k}(v,x) \geq 0.$$

Proof We use induction on m. When $m = 0$, $\tilde{\Delta}_{m,k}(v,x)$ is zero unless $k = 0$ in which case it is 1.

We claim that for $m > 0$

$$\tilde{\Delta}_{m,k}(v,x) = (v+1-x)\tilde{\Delta}_{m-1,k}(v+1,x) + kx\tilde{\Delta}_{m-1,k-1}(v+1-x,x).$$

Then the inductive step is clear because we can assume that $k \geq 0$ and for the given range of v and x, the number $v+1-x$ is also at least 0.

$\square$

Estimating the size of ζ

We have that

$$F_m(s) = \int_0^1 f_m(x) x^{s-2}\, dx$$

where

$$f_m(x) = \sum_{k=0}^{m} \frac{1}{k+1} \Delta_{m,k}(x).$$

Numerical evidence indicates that the function $f_m(x/h_m)$ differs from $x/(e^x - 1)$ by only about h_m/m at any point of $[0, h_m]$ and so we expect the ratio

$$\frac{F_m(s)}{(s-1)G_m(s)}$$

to provide a good approximation to ζ at $s = 1/2 + it$ as long as $\Gamma(s)$ is as large as h_m/m.

We expect the ratio

$$\frac{F_m(s)}{(s-1)G_m(s)}$$

to provide a good approximation to ζ at $s = 1/2 + it$ as long as $\Gamma(s)$ is as large as h_m/m.

This happens if $|t|$ is at most a bit less than $\frac{2}{\pi}\log m$.

Rough calculations indicate that the ratio is not too far from ζ for t all the way up to $\log m$.

There are good reasons to think that $F_m(s)$ does not oscillate significantly for t larger than $\log m$.

The connection with Connes' operator

The Toeplitz matrix

$$L_m = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m} & \frac{1}{m-1} & \ldots & \frac{1}{2} & 1 \end{pmatrix}$$

can be thought of as acting on polynomials $a_0+a_1x+a_2x^2+\cdots+a_mx^m$ rather than sequences $(a_0,\ldots,a_m)$.

It does so by multiplication by the partial sum

$$\sum_{0}^{m} \frac{x^j}{j+1}$$

of the series for $\frac{-\log(1-x)}{x}$ (followed by truncation back to a polynomial of degree m).

In this context the upper triangular matrix U_m maps a polynomial q of degree m to

$$\frac{1}{1-x} \int_x^1 \frac{q(t) - q(0)}{t} \, dt.$$

The operator of Connes is built from a multiplication operator and an integral operator much like these, acting on an infinite-dimensional function space.