COMMENTARY AND COMPARISONS OF SOME APPROACHES TO GRH

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1. EULER PRODUCTS

NO GO: 99% OF THE 'PROOFS' OF RH THAT ARE SUBMITTED TO THE ANNALS (ABOUT 3 PER WEEK) CAN BE REJECTED ON THE BASIS THAT THEY ONLY USE THE FUNCTIONAL EQUATION (F.E.)

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$$

SO THAT THE PROOF WOULD APPLY EQUALLY WELL TO LINEAR COMBINATIONS

$$F(s) := \sum_{j=1}^{n} c_j \cdot L(s, \chi_j), \quad n \geq 2$$

WHERE $L(s, \chi_j)$ ARE L-FUNCTIONS WITH THE SAME F.E.

- IT IS KNOWN (DAVENPORT-HEILBRONN) THAT NON-TRIVIAL LINEAR COMBINATIONS HAVE ZEROS OFF $Re(s) = \frac{1}{2}$.

- ONE EXPECTS THAT SUCH $F(s)$ STILL HAVE 100% OF THEIR ZEROS ON $Re(s) = \frac{1}{2}$;

SELBERG, BOMBIERI/HEJHAL, ..

TO REMEDY THIS NO GO, SELBERG INTRODUCED WHAT IS CALLED TODAY "SELBERG'S CLASS OF EULER PRODUCTS". WHILE INTERESTING FROM AN AXIOMATIC POINT OF VIEW, IT IS NOT INTRINSIC, ESPECIALLY SINCE THE ONLY KNOWN MEMBERS ARE L-FUNCTIONS COMING FROM AUTOMORPHIC FORMS:

$L(s,\pi)$ OR MORE GENERALLY $L(s,\pi,\rho)$

WHERE π IS AN AUTOMORPHIC CUSPIDAL REPRESENTATION OF GL_n AND ρ A FINITE DIMENSIONAL REPRESETATION OF ${}^LG$ $(G=GL_n)$.

- THE $L(s,\pi)$'s ALL HAVE ANALYTIC CONTINUATIONS AND FUNCTIONAL EQUATIONS $s \to 1-s$, $\pi \to \tilde{\pi}$ (THE CONTRAGREDIENT), JACQUET-GODEMENT.

  THEY ALL HAVE AN EXPECTED GRH.

- A KEY POINT IS THAT THERE ARE ONLY COUNTABLY MANY SUCH π'S, SO THAT THE L-FUNCTIONS ARE 'RIGID' (PROBABLY A NECESSARY CONDITION FOR GRH TO HOLD).

- THE F.E. WHICH IS A CONSEQUENCE OF THE ADDITIVE THEORY AND THE EULER PRODUCT WHICH COMES FROM THE MULTIPLICATIVE STRUCTURE ARE DIFFICULT TO STUDY SIMULTANEOUSLY.

2] WHAT IS KNOWN TOWARDS GRH ?

ALL METHODS ARE BASED ON POSITIVITY.

• HADAMARD / DE LA VALLEÉ POUSSIN : $\zeta(s) \neq 0$, $Re(s) = 1$.

$$L(s, \Pi \times \tilde{\Pi}) := \zeta(s) L^2(s, \pi \times \tilde{\pi}) L^2(s+it_0, \pi) L^2(s-it_0, \tilde{\pi}) L(s+2it_0, \pi \times \pi) L(s-2it_0, \tilde{\pi} \times \tilde{\pi})$$

has positive coefficients,

$$\Pi = \mathbb{1} \boxplus \pi \otimes \alpha^{it_0} \boxplus \tilde{\pi} \otimes \alpha^{-it_0}.$$

($L(s, \pi_1 \times \pi_2)$ IS THE RANKIN-SELBERG L-FUNCTION)

$\Rightarrow L(s, \pi) \neq 0$ FOR $Re(s) = 1$; STANDARD ZERO FREE REGIONS IN TERMS OF CONDUCTOR OF π.

• FOR SAY $\zeta(s)$ ONE CAN MAKE SMALL (BUT HARD EARNED) IMPROVEMENTS GIVING UPPER BOUNDS FOR $\zeta(1+it)$ USING VINOGRADOV'S MEAN VALUE ESTIMATES, RECENT PROGRESS ON THE LATTER DOES NOT YIELD MUCH HERE.

EISENSTEIN SERIES AND POSITIVITY:

THIS IS THE MOST POWERFUL METHOD TO DATE ON GRH. THE SPECTRAL THEORY OF EISENSTEIN SERIES FOR GENERAL $G/\mathbb{Q}$ (SELBERG / LANGLANDS) SHOWS THAT THE CONSTANT TERM ALONG A (MAXIMAL) PARABOLIC SUBGROUP HAS NO POLES ON THE UNITARY AXIS; THIS IS A CONSEQUENCE OF THE MAASS-SELBERG INNER PRODUCT.

$\Rightarrow L(s, \pi, \rho) \neq 0$ FOR $Re(s) = 1$ FOR ALL KNOWN CASES JACQUET-SHALIKA, SHAHIDI.

• THE ABOVE CAN BE MADE EFFECTIVE (GELBART – LAPID – SA) AND EVEN ACHIEVE STANDARD ZERO FREE REGIONS (SA, GOLDFELD / LI, HUMPHRIES)

TO ILLUSTRATE WHY THIS METHOD IS MORE POWERFUL AT LEAST AT PRESENT CONSIDER

$$L(s, \pi, \mathrm{sym}^9), \quad \pi \text{ ON } GL_2/\mathbb{Q},$$

$\rho = \mathrm{sym}^9$ OF $GL_2(\mathbb{C})$, EULER PRODUCT OF DEGREE 10.

IT IS NOT KNOWN TO CONVERGE IN $\mathrm{Re}(s) > 1$ (WE DON'T KNOW THE RAMANUJAN CONJECTURES)

YET THE EISENSTEIN SERIES POSITIVITY APPLIED ON $G = E_8$ SHOWS THAT

$$L(s, \pi, \mathrm{sym}^9) \neq 0 \quad \text{FOR} \quad \mathrm{Re}(s) = 1$$

WITH AN EXPLICIT ZERO FREE REGION!

3). POSITIVITY AT THE CENTRAL POINT (EISENSTEIN SERIES)

π SELF DUAL, $\pi = \tilde{\pi}$, $L(s,\pi)$ IS REAL FOR s REAL.

$$GRH \Longrightarrow L\left(\tfrac{1}{2}, \pi\right) \geqslant 0 .$$

- THE VALUE $L(\frac{1}{2},\pi)$ (IF $\varepsilon(\pi)$ ROOT NUMBER IS 1) HAS ARITHMETIC MEANING FOR CERTAIN π's (BIRCH / SWINNERTON DYER CONJECTURES AND GENERALIZATIONS)
- ANOTHER FEATURE AT $s=\frac{1}{2}$ IS THAT L CAN VANISH TO ORDER BIGGER THAN 1 (BSD).

FOR $\pi = \tilde{\pi}$, π IS SYMPLECTIC IF $L(s,\pi,\Lambda^2)$ HAS A POLE AT $S=1$.

- (LAPID-RALLIS) π SYMPLECTIC $\Rightarrow L(\frac{1}{2},\pi) \geqslant 0$ (INCLUDES ALL KNOWN CASES).

AGAIN THIS IS DERIVED FROM A POSITIVITY OF EISENSTEIN SERIES (RESIDUES) ON A CORRESPONDING SYMPLECTIC GROUP (MAASS-SELBERG INNER PRODUCT).

- THE POTENTIAL USE OF EISENSTEIN SERIES FOR SL_2 AND THEIR VALUES AT CM POINTS HAS BEEN EXAMINED SPECTRALLY BY GARRET/BOMBIERI. AND EARLIER DEURING CLASS NUMBERS, PSUEDO CUSP FORMS; HEJHAL, COLIN DE VERDIERE.

4. FUNCTION THEORY (POLYA, ...)

NO GO PROBLEM (FOR ZETA ITSELF ONE MIGHT ARGUE THAT HAMBURGER'S THEOREM RIGIDIFIES THE PROBLEM)

$$\Phi(t) = \sum_{n=1}^{\infty} \left(2n^4\pi^4 e^{9t} - 3n^2\pi e^{5t}\right) \exp\left(-\pi n^2 e^{4t}\right)$$

$$= \Phi(-t) \geq 0 \quad \text{(F.E. RIEMANN)}$$

(THIS LAST POSITIVITY IS VERY RARE FOR AN AUTOMORPHIC L-FUNCTION, IN FACT THERE MAY ONLY BE FINITELY MANY SUCH SA / J. JUNG)

RIEMANN'S ξ FUNCTION IS THE FOURIER TRANFORM

$$\xi(x/2) = 8 \int_0^{\infty} \Phi(t) \cos(xt)\, dt$$

$$\Phi(t) \ll \exp\left(\frac{9|t|}{2} - \pi e^{2|t|} \right), \quad t \in \mathbb{R} \qquad (*)$$

SET $z = -x^2$ AND

$$\xi_1(z) = \frac{1}{8}\xi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k z^k}{k!}$$

A FUNCTION OF ORDER $1/2$

- CSORDAS-VARGA SHOW THAT

RH $\Longleftrightarrow$ REAL ROOTEDNESS OF THE k-SHIFTED JENSEN POLYNOMIALS OF DEGREE n:

$$g_{n,k}(x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_{k+j} x^{j}$$

- GRIFFIN-ONO-ROLEN-ZAGIER HAVE SHOWN RECENTLY THAT FOR n FIXED $g_{n,k}$ IS REAL ROOTED AS $k \to \infty$.

I HAVE NOT SEEN THE DETAILS BUT I EXPECT THAT THIS LARGE k CAN BE PROVED FOR LINEAR COMBINATIONS $F(s)$ AS WELL.

IN ANOTHER DIRECTION ONE CAN TRY DEFORM $\zeta(s)$ INTO A FAMILY OF ENTIRE FUNCTIONS AND FOLLOW THE LOCATION OF THE ZEROS. FOR THE CONSTANT TERM OF EISENSTEIN SERIES THIS WAS DONE PHILLIPS/S DEFORMING IN TEICHMULLER SPACE.

IN THIS CASE THE PRIME NUMBER THEOREM IS STABLE.

dE-BRUIN / NEWMAN DEFORMATION;

$b \in \mathbb{R}$;

$$\Xi_b(z) := \int_{-\infty}^{\infty} \exp(izt - bt^2)\,\Phi(t)\,dt$$

CONVERGES FROM (*), $b=0$ IS RIEMANN'S Ξ.

RESULTS OF POLYA SHOW THAT IF $b<b'$ AND $\Xi_{b'}$ IS REAL ROOTED THEN SO IS Ξ_b. SO THE QUESTION IS WHAT IS THE THRESHOLD b^*.

THE NON VANISHING OF $\zeta(s)$ IN $\mathrm{Re}(s)>1$

$\Rightarrow$ $b^* \geq -1/8$.

RODGERS AND TAO SHOW THAT $b^* \leq 0$ ($RH \Leftrightarrow b^*=0$).

5. FUNCTIONAL ANALYSIS

THE FUNCTIONAL ANALYTIC AND SPECTRAL APPROACHES ARE ALL BASED ON THE ACTION OF DILATIONS (MULTIPLICATION) T_λ, $\lambda \in \mathbb{R}^*$ (OR LARGER ABELIAN GROUPS) ACTING ON VARIOUS FUNCTION SPACES

$$T_\lambda f(x) = f(\lambda x)$$

- BEURLING, NYMAN, BAEZ-DUARTE, ... BURNOL

$$\rho_1(x) = \{1/x\} \quad \in L^2(0,\infty)$$

RH $\Longleftrightarrow$ THE CLOSED CONTRACTION INVARIANT SUBSPACE GENERATED BY (THE SEMI GROUP) $\{T_\lambda \rho_1\}_{\lambda>1}$ CONTAINS $L^2(0,1)$, IN FACT IT SUFFICES THAT IT CONTAIN $\chi_{[0,1]}$.

$$\frac{\zeta(s)}{-s} = \int_0^\infty \rho_1(x)\, x^s \frac{dx}{x} .$$

· VASYUNIN EXAMINED THE PROBLEM OF APPROXIMATING $\chi_{[0,1]}$ WITH LINEAR COMBINATIONS WITH $\lambda = n$, $n \in \mathbb{Z}$, AND TAKING ON BINARY VALUES 0, 1. HE FOUND INFINITE FAMILIES OF SOLUTIONS AS WELL AS SPORADIC ONES — NOT ENOUGH TO GET $\chi_{[0,1]}$.

· BOBER, FOLLOWING AN INSIGHT OF VILLEGAS WHICH RELATES THESE CONSTRUCTIONS OF VASYUNIN TO HYPERGEOMETRICS ${}_nF_{n-1}$ WITH INTEGRAL COEFFICIENTS AND FINITE MONODROMY, SHOWS THAT VASYUNIN'S LIST IS COMPLETE.

WHILE THIS SHOWS THAT ONE CANNOT ACHIEVE THIS APPROXIMATION WITH THESE RESTRICTED LINEAR COMBINATIONS OF FRACTIONAL PARTS, IT IS A BEAUTIFUL CONNECTION TO HYPERGEOMETRICS AND THEIR MONODROMY AND ALSO TO CHEBYSHEV'S ELEMENTARY APPROACH TO THE P.N.TH.

VARIATIONS ON THE T_λ THEME LEAD TO REALIZING THE ZEROS SPECTRALLY AT LEAST AT A FORMAL LEVEL.

- IF WE HAVE A SPACE OF FUNCTIONS (OR DISTRIBUTIONS) WHICH IS INVARIANT UNDER THE GROUP T_λ ($\lambda \in \mathbb{R}^*$) THEN THE EIGEN-FUNCTIONS MUST BE OF THE FORM

$$f(x) = x^\rho \quad , \rho \in \mathbb{C}.$$

THE LINEAR SPACE V ON $\mathbb{R}_{>0}$ OF FUNCTIONS

$$h(x) = \sum_{n \in \mathbb{Z}} f(nx) \ , \ f \in \mathcal{S}(\mathbb{R}), f(0) = \hat{f}(0) = 0.$$

IS T_λ INVARIANT.

THE DISTRIBUTIONS D ANNIHILATING V IS T_λ INVARIANT AND FROM

$$\int_0^\infty \Big(\sum_n f(nx)\Big) x^s \frac{dx}{x} = \zeta(s)\tilde{f}(s)$$

$\Rightarrow$ EIGENFUNCTIONS D OF T_λ CORRESPOND TO $\zeta(\rho) = 0$.

THIS IS THE BASIS OF VARIOUS SPECTRAL INTERPRETATIONS OF THE ZEROS (CONNES, BERRY/KEATING, BENDER/BRODY/MULLER, ...)

(IN THE LATTER TWO THERE IS A SECOND ORDER DIFFERENTIAL OPERATOR THAT COMMUTES WITH T_λ).

- THESE SPECTRAL INTERPRETATIONS STILL HAVE THE NO GO PROBLEM.

- IN THE CASE THAT $\zeta(s)$ OR $L(s,\chi)$ HAS A MULTIPLE ZERO THESE DILATION OPERATORS CANNOT BE DIAGONALIZED, THIS FEATURE IS AN IMPORTANT PHILOSOPHICAL ONE. (*)

- WHY DO WE WANT A SPECTRAL INTERPRETATION? LINEARIZES OUR PROBLEM AND AS LONG AS THE LINEAR SPACE AND OPERATOR HAVE STRUCTURES THAT CAN BE INVESTIGATED THIS COULD BE USEFUL.

- THE 'HILBERT-POLYA' IDEA THAT THIS LINEAR SPACE COMES WITH AN INNER PRODUCT MAKING THE OPERATOR SELF ADJOINT IS I THINK NAIVE. IT HAS NOT WORKED IN OTHER SETTINGS AND DOES NOT ALLOW FOR (*)

6. FUNCTION FIELDS (ARTIN, WEIL, ...)

CURVES C OVER FINITE FIELDS. THERE ARE TWO STEPS:

(i) SPECTRAL INTERPRETATION OF ZEROS THROUGH THE ACTION OF FROBENIUS ON COHOMOLOGY.

(ii) PROOF OF RH THROUGH POSITIVITY.

WEIL GAVE TWO PROOFS

(a) GENERALIZING HASSE'S $g=1$ CASE, PASSING TO JAC(C) AND ANALYSING FROBENIUS ϕ IN END(JAC), ROSATI INVOLUTION $'$, TRACE$(\phi'\phi) > 0$.

(b) VIA CORRESPONDECES ON $C \times C$. THE POSITIVITY COMING FROM CASTELNOUVO'S INEQUALITY ON SURFACES. BOMBIERI (1996) GIVES AN ACCOUNT WITH POINTERS TO THE EXPLICIT FORMULA (BELOW). A VARIATION OF THIS PROOF YIELDS THE POSITIVITY DIRECTLY FROM RIEMANN-ROCH ON $C \times C$ (MATTUCK-TATE).

(c) STEPANOV'S ELEMENTARY PROOF; USES WHAT COMBINATORISTS CALL TODAY "THE POLYNOMIAL METHOD". A FEATURE OF THIS PROOF IS THAT IT GOES BEYOND THE RH – GIVING PRECIOUS INFORMATION FOR $g > \sqrt{p}$, $g = g(C)$, $\mathbb{F}_p$.

(d). DELIGNE'S PROOF USING FAMILIES, MONODROMY, HIGH TENSOR POWER REPRESENTATIONS, POSITIVITY FROM THERE. AS YET THIS IS THE ONLY METHOD THAT HAS SUCCEEDED FOR HIGHER DIMENSIONAL VARIETIES.

(B1)

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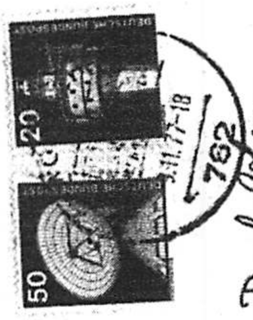

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We haven't proved the RH yet.

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7) SPECTRAL INTERPRETATION VIA $\mathbb{A}/\mathbb{Q}^*$ (COHEN, CONNES)

(NOT OBVIOUSLY NO GO)

$\mathbb{A}$ RING OF ADELES OF $\mathbb{Q}$ (ADDITIVE) $(a_\infty, a_2, a_3, \dots)$ $a_p \in \mathbb{Z}_p$ for a.a. p.

$\mathbb{A}^*$ RING OF IDELES multiplicative $(a_\infty, a_2, a_3, \dots)$ $a_v \neq 0$, $a_p \in U_p$ for a.a. p.

USUAL SPACES (TATE 'VALUATION VECTORS')

$\mathbb{A}/\mathbb{Q}$ ADDITIVE ACTION, $\mathbb{A}^*/\mathbb{Q}^*$ MULT. ACTION.

$\mathbb{A}/\mathbb{Q}^*$ VERY SINGULAR.

IN MY FIRST MEETINGS WITH PAUL AS A GRADUATE STUDENT (1977) HE POINTED TO $\mathbb{A}/\mathbb{Q}^*$ AND THE ACTION $T_y : x \to xy$, $y \in \mathbb{A}^*/\mathbb{Q}^*$, $x \in \mathbb{A}/\mathbb{Q}^*$ (MULTIPLICATION OVER ADDITION) AND THAT THE TRACE OF THIS ACTION LEADS FORMALLY TO THE RHS OF THE RIEMANN-GUINAND-WEIL EXPLICIT FORMULA.

LHS = SUM OVER ZEROS AND POLES OF ALL HECKE L-FUNCTIONS OF K (A NO' FIELD) OF FOURIER TRANSFORM OF h

$$\text{RHS} = \sum_v \int'_{K_v^*} \frac{h(u^{-1})}{|1-u|} d^*u.$$

THE TECHNICALITIES AROUND THE VERY SINGULAR SPACE $\mathbb{A}/\mathbb{Q}^*$ LED PAUL TO STUDY SELBERG'S WORK ON THE TRACE FORMULA (THIS BECAME MY THESIS TOPIC). COHEN REALIZED EARLY ON THAT TO MAKE ANY USE OF THIS ONE HAS TO GO BEYOND FORMALITIES IN SEARCHING FOR THE SOURCE OF POSITIVITY. HIS THINKING WAS CERTAINLY INFLUENCED BY WEIL'S CORRESPONDENCE PROOF ON C×C AND WEIL'S QUADRATIC FUNCTIONAL POSITIVITY EQUIVALENCE IN HIS EXPLICIT FORMULA (BOMBIERI HAS MADE AN IN DEPTH STUDY OF THE LAST)

OVER THE YEARS HE (PAUL) TRIED MANY VARIANTS FOR THE SPACE $\mathbb{A}/\mathbb{Q}^*$ MOSTLY COMBINATORIAL, SOME OF WHICH MADE CONTACT WITH SIEVE THEORY.

CONNES (1999) PUT FORTH A PRECISE INTERPRETATION IN TERMS OF THIS $\mathbb{A}^*/\mathbb{Q}^*$ ACTION AND ITS TRACE. THE SOBOLEV SPACES THAT HE USED, DO NOT ALLOW FOR ZEROS OFF THE LINE, NOR FOR HIGH MULTIPLICITY ZEROS.

THIS WAS RECTIFIED BY MEYER (2005) WHO ALLOWS MORE GENERAL TOPOLOGICAL VECTOR SPACES. HE REALIZES THE EXPLICIT FORMULA AS THE TRACE OF THE ACTION OF $\mathbb{A}^*/\mathbb{Q}^*$ ON A SUITABLE SPACE OF FUNCTIONS ON $\mathbb{A}^*/\mathbb{Q}^*$.

THE ZEROS (PLUS POLE) ARE THE EIGENVALUES OF THIS ACTION.

MY BELIEF IS THAT THE METHOD OF FAMILIES OF L-FUNCTIONS IS THE WAY FORWARD (AS IT IS FOR DELIGNE IN THE GENERAL WEIL CONJECTURES). IT IS A GENERAL PRINCIPLE, THAT IT IS DIFFICULT TO STUDY AN ISOLATED OBJECT AND ONE WOULD LIKE TO DEFORM IT INTO OBJECTS OF A SIMILAR GENUS.

IN THE CASE OF AUTOMORPHIC L-FUNCTIONS, TEMPLIER/SHIN/SA HAVE PUT FORTH A DEFINITION OF A GENERAL FAMILY. THESE SERVE TO FORMULATE BASIC CONCEPTS ASSOCIATED WITH THE SYMMETRY OF A FAMILY AND TO ASK AND ANSWER VARIOUS QUESTIONS FOR FAMILIES OF L-FUNCTIONS.

WHILE NONE OF THESE CAN BE SAID TO BE TOOLS FOR PROVING GRH THE RESULTS THAT CAN BE PROVED ARE OFTEN GOOD ENOUGH TO BE COMPLETE SUBSTITUTES OF GRH IN CERTAIN APPLICATIONS (THE BOMBIERI/VINOGRADOV THEOREM IS A PROTO-TYPICAL EXAMPLE).

WHAT IS LACKING IS A GENUINE MARRAIGE OR GLUE FOR THE L-FUNCTIONS IN A FAMILY, AS THERE IS VIA GROTHENDIECK'S THEORY AND MONODROMY IN THE FUNCTION FIELD.

TO POINT TO THE FIRST ROAD BLOCK: WHAT IS THE ANALOGUE OF THE SYMMETRIC POWER L-FUNCTIONS OF SATO AND LATER LANGLANDS, FOR FAMILIES OF AUTOMORPHIC L-FUNCTIONS?