# Positivity of L-functions and "Completion of square"

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#### Outline

- Riemann hypothesis
- Positivity of L-functions
- 3 Completion of square
- Positivity on surfaces

# Riemann Hypothesis (RH)

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

$$= \prod_{p, \text{ primes}} \frac{1}{1 - p^{-s}}, \quad s \in \mathbb{C}, \text{ Re}(s) > 1$$

Analytic continuation and Functional equation

$$\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1-s), \quad s \in \mathbb{C}$$

#### Conjecture

The non-trivial zeros of the Riemann zeta function  $\zeta(s)$  lie on the line

$$\operatorname{Re}(s) = \frac{1}{2}$$

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# Equivalent statements of RH

Let

$$\pi(X) = \#\{\text{primes numbers } p \leq X\}.$$

Then

$$RH \Longleftrightarrow \left| \pi(X) - \int_2^X \frac{dt}{\log t} \right| = O(X^{1/2 + \epsilon}).$$

Let

$$\theta(X) = \sum_{p \in Y} \log p.$$

Then

$$RH \iff |\theta(X) - X| = O(X^{1/2 + \epsilon}).$$

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# RH for a curve C over a finite field $\mathbb{F}_q$

#### Theorem (Weil)

$$|\#C(\mathbb{F}_{q^n}) - (1+q^n)| \leq 2 g_C \sqrt{q^n}.$$

#### Remark

To compare with the case for Q:

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# RH for a curve C over a finite field $\mathbb{F}_q$

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#### More L-functions

#### Example

To an elliptic curve over  $\mathbb Q$ 

$$E: y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Hasse-Weil associates an L-function

$$L(s, E) = \prod_{p: good} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

where, for good p

$$a_p = p + 1 - E(\mathbb{F}_p).$$

# Grand Riemann Hypothesis (GRH)

There are many other L-functions, e.g., those attached automorphic representations on GL(N).

#### Conjecture

Nontrivial zeros of all automorphic L-functions lie on the line

$$\operatorname{Re}(s)=\frac{1}{2}.$$

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# A corollary to Riemann Hypothesis

Suppose that an *L*-function has the following properties

- *L*(*s*) is an entire function.
- $L(s) \in \mathbb{R}$  if s is real.
- L(s) > 0 as  $s \in \mathbb{R}$  and  $s \to \infty$ .

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$$GRH \Longrightarrow L(1/2) \ge 0$$
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# Superpositivity: non-leading terms

#### Lemma (Stark–Zagier (1980), Yun–Zhang)

Let  $\pi$  be a self-dual cuspidal automorphic representation of  $GL_n$ . Normalize its functional equation such that

$$L(s,\pi)=\pm L(1-s,\pi).$$

Then

$$GRH \Longrightarrow L^{(r)}(1/2,\pi) \geq 0$$
, for all  $r \geq 0$ .

Here

$$L(s,\pi) = \sum_{r=0}^{\infty} L^{(r)}(1/2,\pi) \frac{(s-1/2)^r}{r!}.$$

#### The idea of proof

Hadamard product expansion (and the functional equation and the self-duality)

$$L(s+1/2)=c\cdot s^r\prod_{\rho}\left(1-\frac{s^2}{\rho^2}\right),$$

- the product runs over all the zeros  $\frac{1}{2} \pm \rho$  of L(s) such that  $\rho \neq 0$ ,
- r = ord<sub>s=1/2</sub>L(s), and c > 0 is the leading Taylor coefficient.

Now note that

$$GRH \iff \operatorname{Re}(\rho) = 0.$$



# Super-postivity of L-functions

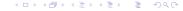
- Super-postivity does not imply GRH. But it implies the non-existence of Landau–Siegel zero.
- Known for Riemann zeta function (Polya, 1927). Sarnak introduced a notion of "positive definite" for L-functions. If an L-function is positive definite then it is "super-positive". Not known if there are infinitely many positive definite L-functions
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# Completion of square

The super-positivity suggests us

to express  $L^{(r)}(1/2,\pi)$  in terms of some "squared quantity".

We explain two such examples

(Gross–Zagier, Yuan–Zhang–Zhang) The first derivative

$$L'(1/2,\pi)\geq 0$$

if  $\pi$  appears in the cohomology of Shimura curve over a (totally real) number field F.

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# Gross-Zagier formula

#### Theorem

Let E be an elliptic curve over  $\mathbb{Q}$ . There is a point  $P \in E(\mathbb{Q})$  such that

$$L'(1,E)=c\cdot\langle P,P\rangle,$$

where the RHS is the Néron-Tate height pairing

$$\langle \cdot, \cdot \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$

and c is a positive number.

The point *P* in the above formula is the so-called Heegner point. The Néron–Tate height pairing is known to be positive definite. Hence

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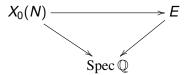
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# Heegner points on modular curve

• The modular curve  $X_0(N)$  is moduli space classifying elliptic curves with auxiliary structure:



 The Heegner points are represented by those special elliptic curves with complex multiplication.

#### **Drinfeld Shtukas**

Now fix  $k = \mathbb{F}_q$ , and X/k a smooth geometrically connected curve. We consider the moduli stack of Drinfeld Shtukas of rank n. For a k-scheme S, we have

$$\operatorname{Sht}^r_{\operatorname{GL}_n,X}(S) = \left\{ \begin{array}{l} \text{vector bundles } \mathcal{E} \text{ of rank } n \text{ on } X \times S \\ \text{with minimal modification } \mathcal{E} \to (\operatorname{id} \times \operatorname{Frob}_S)^* \mathcal{E} \\ \text{at } r\text{-marked points } x_i : S \to X, 1 \leq i \leq r \end{array} \right\}$$

We have

$$X^{r} = \underbrace{X \times_{\operatorname{Spec}k} \cdots \times_{\operatorname{Spec}k} X}_{r \text{ times}}$$

# Higher Gross-Zagier formula

#### Theorem (Yun-Zhang)

Fix  $r \in \mathbb{Z}_{\geq 0}$ . Let E be a semistable elliptic curve over k(X). Then there is an algebraic cycle (the Heegner–Drinfeld cycle) on  $\operatorname{Sht}^r_{\operatorname{PGL}_2,X}$  such that the E-isotypic component of the cycle class  $Z_{r,E}$  satisfies

$$L^{(r)}(1,E)=c\cdot \Big(Z_{r,E},Z_{r,E}\Big),$$

where  $(\cdot, \cdot)$  is the intersection pairing.

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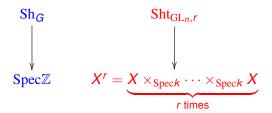
#### Comparison with the number field case

In the number field case, the analogous spaces only exist when  $r \leq 1$ .

• When r = 0, this is the double-coset space

$$G(F)\setminus (G(\mathbb{A})/K)$$
.

When r = 1, the analogous space is Shimura variety



In the function field case, we need not restrict ourselves to the leading coefficient in the Taylor expansion of the L-functions.

#### Question

In the number field case, should there be any geometric interpretation of the non-leading coefficients, for example,  $L^{(r)}(1,E)$  when E is an elliptic curve over  $\mathbb{Q}$ ?

Recall that the conjecture of Birch and Swinnerton-Dyer gives a geometric interpretation of the leading term

$$L^{(r)}(1, E) = c \cdot \operatorname{Reg}_{F} \cdot \operatorname{III}_{E}$$



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# Intersection pairing on an algebraic surface

S: smooth projective surface over a field k. Div(S): free abelian group of divisors on S. There is an intersection pairing

$$\begin{split} \operatorname{Div}(S) \times \operatorname{Div}(S) &\to \mathbb{Z} \\ (C,D) &\mapsto C \cdot D \end{split}$$

# Hodge index theorem for a surface

#### Theorem

Let S be a surface over a field k.

If H is an ample divisor, and  $D \cdot H = 0$ , then

$$D \cdot D \leq 0$$
.

NS(S) = Div(S) modulo numerical equivalence. Then the index of the intersection matrices of a basis of NS(S) is

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#### Weil's proof of RH for curves

Consider a curve  $X/\mathbb{F}_q$ , and the surface

$$S = X \times_{\operatorname{Spec}\mathbb{F}_q} X$$

Compute the intersection matrix of 4 divisors

$$pt \times X$$
,  $X \times pt$ ,  $\Delta$ ,  $F$ 

F is the graph of the Frobenius

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Denote  $N = X(\mathbb{F}_q)$ . The intersection matrix

$$T = \left(\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & q \\ 1 & 1 & 2 - 2g & N \\ 1 & q & N & q(2 - 2g) \end{array}\right)$$

$$H = pt \times X + X \times pt$$
 ample

$$\implies det(T) = (N - (1+q))^2 - 4qg^2 \le 0$$

$$\Longrightarrow |N-(1+q)| \le 2g\sqrt{q}$$
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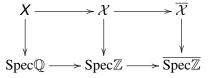
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### Arithmetic surface

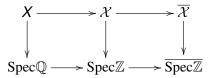
An arithmetic surface  $\overline{\mathcal{X}}$  is the data of a relative curve  $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$  with a metric on the Riemann surface  $X(\mathbb{C})$ .



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## Hode index theorem for arithmetic surface

### Theorem (Faltings, Hriljac)

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If  $\overline{H}$  is an ample divisor, and  $\overline{D} \cdot \overline{H} = 0$ , then

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#### Remark

- This positivity together with Gross–Zagier formula implies  $L'(1, E) \ge 0$ . (in addition to RH over finite fields)
- Comparison the proof of  $L'(1, E) \ge 0$  with the proof of RH for curve over a finite field. The geometric ingredients in them seem to be the best evidence to RH.

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# Yuan's proof of Hodge index for arithmetic surfaces

Yuan: an arithmetic line bundle  $\overline{\mathcal{L}}\mapsto$  a convex body in  $\mathbb{R}^2$ .

### Lemma (Brunn-Minkowski)

Let A, B be two compact subsets of  $\mathbb{R}^n$ , and let A+B denote the Minkowski sum

$$A + B = \{a + b : a \in A, b \in B\} \subset \mathbb{R}^n$$
.

Then

$$vol(A + B)^{1/n} \ge vol(A)^{1/n} + vol(B)^{1/n}$$
.

## Surfaces

- The first kind is a surface over a filed k, e.g.  $C \times C$  for a curve C over k.
- The second kind is arithmetic surface: its base is an arithmetic curve Spec and its fiber are curves over fields.
- The third kind is unknown: "Spec $\mathbb{Z} \times_{\operatorname{Spec}\mathbb{F}_1} \operatorname{Spec}\mathbb{Z}$ "? It should be a fibration with its base an arithmetic curve  $\operatorname{Spec}\mathbb{Z}$  and with fibers also being arithmetic curves.
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# The third example: ABC and Landau-Seigel zeros

#### Definition

A Landau–Siegel zero is a zero  $\beta$  of  $L(s, \chi_d)$  (for the quadratic character  $\chi_d$  associated to  $\mathbb{Q}[\sqrt{d}]$ ) lying in

$$[1 - c/\log|d|, 1]$$

for a small c > 0.

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# Faltings heights of CM abelian varieties

The key to the theorem of Granville–Stark is Kronecker limit formula for an imaginary quadratic field  $K = \mathbb{Q}[\sqrt{-d}]$ . This formula relates the Faltings height of an elliptic curve  $E_d$  with complex multiplication by  $O_K$  to L-function

$$h_{\text{Fal}}(E_d) = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log |d|.$$

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