RH: Problem Session

June 6, 2018.

Question 1

Define what to count such that

$$\#\{\text{things } \leq X\} \sim \frac{f(X)}{L(s)}.$$

The left hand side should be finite so that $L(s) = 0$ yields a contradiction. See Conjecture 7.1 in the paper of Young [2].

Question 2

Prove that the geometric Frobenius on the cohomology is diagonalisable (for any projective, smooth variety).

Question 3

Is it true that a method that gives $\langle |L(\frac{1}{2})|^k \rangle$ generalises to give

$$\langle L(\frac{1}{2} + \alpha_1) \cdots L(\frac{1}{2} + \alpha_k) \rangle?$$

Comments:

- Yes, if using approximate functional equation
- Maybe no if α not small
- No if using the period formula.

Question 4

If we look at $L(\frac{1}{2}, \chi_d)$, it has a different distribution to $L(\frac{1}{2} + it, \chi_d)$, for $t \neq 0$. Characterise how the distribution of the latter changes as t grows.

Comments:

- Looked at by Keating and Odgers in RMT case, see [1].

Question 5

If we write that $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + S(t)$, then is there a functional inverse for $N^{-1}(T)$? Preferable for it to be 'more analytic' than the Lambert W function.

Question 6

Thanks to the Weil conjectures, it is known that

$$\sum_{x\in\mathbb{F}_p\setminus\{0,1\}} e_p\left(x + \frac{1}{x} + \frac{1}{x-1}\right) = O(\sqrt{p}).$$

But is it possible to establish (preferably via an analytic method) even the weaker bound

$$\sum_{x\in\mathbb{F}_p\setminus\{0,1\}} e_p\left(x + \frac{1}{x} + \frac{1}{x-1}\right) = o(p)$$

without going through the Weil conjectures (in particular, without having to establish rationality of the relevant zeta function)?

Question 7

Investigate the relationship between $\zeta(\frac{1}{2} + it_0)$ large and zeros very near t_0. Justify the bound

$$\exp\left(\sqrt{\tfrac{1}{2}\log T\log\log T}\right)$$

on the maximum size of $\zeta(\frac{1}{2} + it)$ up to height T via a conjecture on zeros.

Comments:

- Compare to $S(t)$ (see Question 6).
- Maybe better to look at big gaps between zeros.

Question 8

Give decent bounds for how often we get multiple (high-order) zeta-zeros.

Question 8 1/2

Find a universal bound on the multiplicity for any primitive L-function as a function of the degree.

Question 9

Provide convincing evidence of the existence of an elliptic curve over $\mathbb{Q}$ of rank at least 30.
By convincing evidence, we mean enough to convince Michael Rubinstein. It might include exhibiting an elliptic curve with at least 30 independent points of infinite order, or a Dirichlet series that appears to have the characteristics of the L-function of an elliptic curve over $\mathbb{Q}$ of analytic rank 30 or higher.
The prize from Michael Rubinstein for doing so is $1000 USD.

Question 10

Take the Laplacian Δ on $\mathbb{H}/\Gamma(1)$ modular surface. Look at rank$\geq$ 1 extension. Can one explain the connection, if it exists, to the Berry-Keating operator?

Question 11

Classify all pairs of sets of roots of unity that interlace and the Galois conjugates also interlace. An approach that generalises is desired. Might be related to algebraic hypergeometric functions.

Question 12

Characterise geometric vs harmonic families of L-functions, in the sense of those mentioned in Peter Sarnak's and Will Sawin's talks.

Comments:

- Maybe sub-main terms
- Some orthogonal families have discrete critical values
- Different than algebraic vs transcendental
- Maybe look at Rankin-Selberg?

Question 13

Fix some data in the functional equation, for example the conductor. Are there non-hypergeometric L-functions with the same functional equation?

Comments:

- Likely
- The special case of the Hodge vector $(1, \ldots, 1)$ may be different (and likely), whereas other vectors might be less likely (and will be harder).

Question 14

Find an integral representation for ζ over some matrix group,

$$\zeta(s) = \underbrace{\int \cdots \int}_{\text{matrix group}} f(s) d\mu_{\text{Haar}},$$

where f is elementary, natural.

Comments:

- Unlikely
- Maybe if the matrix group was infinite?

Question 15

Take an Eisenstein series $E(z,s)$, $z \in \mathbb{H}$. Want to know about zeros as a function of s - are almost all on $\sigma = 1/2$? What about for linear combinations?

Comments:

- There is a positive proportion in some cases.
- Maybe see Hejhal's ICM paper.

Question 16

(Without cheating) construct a function which provably has RMT zero statistics.

Question 17

The main term of the moments satisfy Painlevé equations - what are the arithmetic implications of this?

Question 18

Prize: \$1000, see Brian Conrey. Assuming GRH prove that there exists a c such that

$$\int_0^T |\zeta(1/2 + it)| dt \sim cT(\log T)^{1/4}.$$

Question 19

Take $E/\mathbb{Q}$ an elliptic curve. Then

$$\frac{1}{\log\log X}\sum_{p\leq X}\frac{a_p}{p}\sim\tfrac{1}{2}-\text{rank}.$$

Make this practical/effective. Maybe one has to average differently?

Question 20

Same setting as Question 15, look at

$$\sum\alpha_j E(z_j,s),\ \text{where}\ \sum\frac{\alpha_j E(z_j,s)}{\zeta(s)}$$

has no pole. Is $\zeta(s)$ the only L-function for which this is true?

Question 21

Prove that if $L(s)$ has a pole at $s=1$ then $L(s)/\zeta(s)$ is an L-function.

References

[1] B. E. Odgers, J. P. Keating, Symmetry Transitions in Random Matrix Theory & L-functions, *Comm. Math. Phys.*, **281**(2), 499–528, 2008.

[2] M. P. Young, Moments of the critical values of families of elliptic curves, with applications, *arXiv preprint arXiv:0708.4042*, 2007.