Rubel's Problem: from Hayman's List to the Chabauty Method

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Walter Hayman's list *Research Problems in Function Theory* has been updated for its 50th anniversary. Problem 4.27, posed by Lee Rubel, was related to the famous Prouhet-Tarry-Escott problem. Some modern tools of computational number theory can be used to find explicit counterexamples.

Walter Hayman's list

Research Problems in Function Theory [9] is a collection of problems collated by Walter Hayman in 1967 and fondly known as *Hayman's list*. It gave a big impetus to the development of geometric function theory, the part of complex analysis concerned with the geometric properties of analytic functions.


Walter Hayman.
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Walter loved to encourage younger mathematicians. He and his wife Margaret founded the British Mathematical Olympiad, and the mathematics genealogy website lists his 20 PhD students and 133 mathematical descendants. His own research was groundbreaking, particularly his work on the Bieberbach conjecture and in the area of Nevanlinna theory, which relates the growth and covering properties of meromorphic functions. He was awarded the LMS De Morgan medal in 1995. Fifty years after the publication of the list, Walter Hayman teamed up with Eleanor Lingham (the editor of this newsletter) to gather the state of the art on each of the problems. The resulting *Fiftieth Anniversary Edition* [10] was published in 2019. We were sorry to learn that Walter Hayman passed away on January 1st 2020.

Rubel's problem

The fourth chapter of Hayman's list is about polynomials. Among the problems there, number 4.27 stands out for the strange reason that it is not a problem in function theory but rather a problem in number theory. It was posed by the American analyst Lee Rubel (1927–1995), whose work spanned differential equations, approximation theory, and the theory of analog computing. Among many other surprising results, Rubel showed that there is a single entire function whose derivatives are dense in the space of entire functions with respect to the topology of locally uniform convergence. Here is problem 4.27, as it appears in [9]:

> *4.27 (Lee Rubel). Let $f(x)$ be a real polynomial of degree n in the real variable x such that $f(x) = 0$ has n distinct (real) rational roots. Does there necessarily exist a (real) non-zero number t such that $f(x) - t = 0$ has n distinct (real) rational roots?*
> *(I can prove this for $n = 1, 2, 3$.)*

You might like to try the case $n = 3$ for yourself.

The phrasing suggests Rubel hoped for an affirmative answer, extrapolating from the low-degree cases.

In a straw poll of our local number theorists at the University of Bristol, all had the opposite intuition to Rubel's. We'll give a calculation-free proof below that the answer is *no*, at least for even degrees $n \geq 8$. What is more interesting is to try to exhibit explicit counterexamples. For each degree $n \geq 4$ we pose the following challenge:

> Problem $\texttt{Rubel}(n)$:
> Exhibit $f \in \mathbb{Q}[x]$ of degree n such that $f - t$ has n distinct rational roots if and only if $t = 0$.

If f is a polynomial of degree n, $t \neq 0$, and both f and $f - t$ have n distinct rational roots, then we can rescale the roots of f and $f - t$ to obtain an ideal solution of the *Prouhet-Tarry-Escott problem* (see inset). Those are hard to find, but this doesn't mean it is easy to prove that any particular polynomial f is a solution of $\texttt{Rubel}(n)$.

Prouhet–Tarry–Escott problem

Given $k, n \in \mathbb{N}$, find two distinct sets of integers A and B, both of size n, such that

$$\sum_{a\in A} a^i = \sum_{b\in B} b^i \quad \text{for } i = 1,\ldots,k.$$

A simple appplication of the pigeonhole principle shows that solutions exist whenever $n > k(k+1)/2$. A solution is called *ideal* when $n = k+1$. In that case we have

$$\prod_{a\in A}(x-a) - \prod_{b\in B}(x-b) = c$$

for some non-zero constant c. The largest k for which an ideal solution is known is 11. This solution was found in 1999 by Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen:

$$A = \{\pm 35, \pm 47, \pm 94, \pm 121, \pm 146, \pm 148\},$$

$$B = \{\pm 22, \pm 61, \pm 86, \pm 127, \pm 140, \pm 151\}.$$

For this example, we have

$$\begin{aligned} c &= 6744029455596760054016000 \\ &= 2^{12}.3^9.5^3.7^2.11^2.13^2.17.19.23.29.31. \end{aligned}$$

If you can prove that there is no ideal solution of Prouhet-Tarry-Escott with $k \geq 12$, then you will also have proved that every polynomial of degree $n \geq 13$ with n distinct rational roots is a solution of $\texttt{Rubel}(n)$. See Borwein [5] for more information about the Prouhet-Tarry-Escott problem.

No local obstructions

It is easy to write down an $f \in \mathbb{Q}[x]$ of degree $n \geq 3$ for which there is no $t \in \mathbb{Q}$ such that $f - t$ has n distinct roots. For instance, this is true for $f = x^3 + x$, because it maps $\mathbb{R}$ to $\mathbb{R}$ injectively. For another example, $f = x^3 - 2x$ is an injective mapping from $\mathbb{Q}$ to $\mathbb{Q}$. If $x \neq y$, but $f(x) = f(y)$, then $x^2 + xy + y^2 = 2$, but for $x, y \in \mathbb{Q}$, not both zero, the valuation of x^2+xy+y^2 at 2 is always even. These are both examples of *local obstructions*. In the first case $f - t$ cannot have two distinct real roots, and in the second it cannot have two distinct 2-adic roots.

Ruling out repeated roots stops Rubel's problem from being trivial. For example $f = x(x-1)^2(x+1)^2$ has a local minimum at 1 and a local maximum at -1, so $f - t$ has at most three real roots if $t \neq 0$.

The constraint that f has n distinct rational roots makes the problem a lot more fun. It implies that for $K = \mathbb{R}$ or $K = \mathbb{Q}_p$, every sufficiently small perturbation $\tilde{f}$ of f still factors completely over K. This is because each root of f can be perturbed to a nearby root of $\tilde{f}$ in K. This is shown in the p-adic case by Hensel's lemma. So we cannot hope to solve $\texttt{Rubel}(n)$ by finding a local obstruction.

A non-constructive solution of Rubel's problem

We will answer Rubel's problem by showing that for each $n \geq 4$, there exists a solution to $\texttt{Rubel}(2n)$.

Choose rationals $0 < a_1 < \cdots < a_n$ such that the polynomial $f = \prod_{i=1}^n (x - a_i^2)$ has $n-1$ distinct critical values. This condition holds generically in $\mathbb{A}^n(\mathbb{Q})$.

Case 1: f is a solution of $\texttt{Rubel}(n)$.
Then $F := f(x^2)$ is a solution of $\texttt{Rubel}(2n)$, since it has the $2n$ distinct roots $\pm a_1, \ldots, \pm a_n$. However, if $F - t$ has $2n$ distinct rational roots for some nonzero t, then $f - t$ must have n distinct rational roots, contrary to the hypothesis.

Case 2: f is not a solution of $\texttt{Rubel}(n)$.
Define $G(x,w) = (f(x) - f(w))/(x-w)$. Thinking of G as a polynomial in x, its coefficients are polynomials in w, so its discriminant h belongs to $\mathbb{Q}[w]$. The roots of h in $\mathbb{C}$ are the $n-2$ non-critical preimages under f of each of the $n-1$ critical values of f. Each of these is a simple root of h. So h has degree $d = (n-1)(n-2)$ and has no repeated complex root. In particular, $d \geq 6$, and the discriminant of h is non-zero.

If $f - t$ has rational roots $q_1 < \cdots < q_n$, we have $2n$ rational points on the hyperelliptic curve $Y^2 = h(X)$, given by

$$\left(q_k, \pm \prod_{\substack{i<j \\ i,j\neq k}} (q_i - q_j)\right), \quad k = 1,\ldots,n.$$

Since $\operatorname{disc}(h) \neq 0$, this is a smooth affine curve over $\mathbb{Q}$ of genus $g = \lceil (d-2)/2 \rceil = (n-1)(n-2)/2 - 1$. Although its closure in the projective plane is not smooth, it can be embedded in a smooth curve in projective 3-space, by a map that takes rational points to rational points. Faltings' theorem says that any smooth projective curve of genus at least 2 has only finitely many rational points (see inset). It follows that the curve $Y^2 = h(X)$ has only finitely many rational points.

We deduce that there are only finitely many rational values of t for which $f - t$ has n distinct rational roots. Let $t_1 < \cdots < t_m$ be the complete sorted list of these rational values. Since f is not a solution of $\mathtt{Rubel}(n)$, $m \geq 2$. Consider the degree $2n$ polynomial

$$F := (f - t_1)(f - t_2)\,.$$

By construction F has $2n$ distinct rational roots. We claim F is a solution of $\mathtt{Rubel}(2n)$. Suppose $F - t$ has $2n$ distinct rational roots for some $t \in \mathbb{Q}\setminus\{0\}$. Let α_1 and α_2 be the roots of the quadratic $(x-t_1)(x-t_2)-t$, so that $\alpha_1 + \alpha_2 = t_1 + t_2$ and

$$F - t = (f - \alpha_1)(f - \alpha_2)\,.$$

At most n of the roots of $F - t$ can be roots of $f - \alpha_i$, for $i = 1, 2$, so exactly n are roots of each. In particular α_1 and α_2 are rational and they belong to $\{t_3, \ldots, t_m\}$. But because of the sorting this contradicts $\alpha_1 + \alpha_2 = t_1 + t_2$. □

This proof was simple enough, but we used the big hammer of Faltings' theorem to crack the nut! And we haven't yet written down any explicit counterexample to Rubel's problem.

A sextic f such that $f - t$ has six rational roots for infinitely many t

Define four rational functions of u:

$$A = \frac{2u^2 + 6u + 1}{u^2 + u + 1}, \quad B = \frac{3u^2 + 2u - 2}{u^2 + u + 1},$$

$$C = \frac{u^2 - 4u - 3}{u^2 + u + 1}, \text{ and } R = (ABC)^2\,.$$

Then

$$x^6 - 14x^4 + 49x^2 - R = (x^2 - A^2)(x^2 - B^2)(x^2 - C^2)\,.$$

Our argument using Faltings' theorem does not apply to this case because for each choice of u the critical values of this sextic polynomial in x are repeated.

Faltings' theorem

In 1922 Louis Mordell proved the seminal result that the abelian group of rational points on any elliptic curve defined over $\mathbb{Q}$ is finitely generated. At the end of same paper, he conjectured a restricted version of what became known as the *Mordell conjecture*, that any smooth algebraic curve defined over $\mathbb{Q}$ with genus at least 2 has only finitely many rational points.
The Mordell conjecture was proved by Gerd Faltings in 1983, and he won a Fields Medal in 1986 for this work. A few years later, Paul Vojta gave a very different proof using Diophantine approximation and Arakelov intersection theory, and Enrico Bombieri soon gave a more elementary version of this proof.
Faltings' theorem is *ineffective*: it does not tell us how to enumerate all of the rational points. Bombieri's proof in principle gives a computable bound on how many rational points a curve has but no bound on how much ink it takes to write the points down.

An explicit example for $\mathtt{Rubel}(7)$

Consider the polynomial

$$f = (x-4)(x-3)(x-1)x(x+1)(x+3)(x+4)\,.$$

Because f is odd, the discriminant of $f - t$ with respect to x is an even polynomial in t. We write $\operatorname{disc}(f - t) = h(t^2)$, where h is the irreducible cubic

$$\begin{aligned} h = &-823543\,t^3 + 353645809920\,t^2 \\ &- 35639879984676864\,t \\ &+ 951446985611673600000. \end{aligned}$$

The sextic $h(t^2)$ has six real roots, which are the critical values of f. Any $t \in \mathbb{Q}$ for which $f - t$ has seven roots $q_1, \ldots, q_7 \in \mathbb{Q}$ gives us a rational point

$$(X, Y) = \left(t, \prod_{1 \leq i < j \leq 7} (q_i - q_j)\right)$$

on the affine curve $\mathscr{C} : Y^2 = h(X^2)$.

The curve $\mathscr{C}$ is called *bi-elliptic* because it has nonconstant maps to two different elliptic curves. First, $\mathscr{C}$ covers the elliptic curve $E_1 : Y^2 = h(X)$, by the map $(X, Y) \mapsto (X^2, Y)$. This map does not help us, because the point of $\mathscr{C}$ coming from $t = 0$ has infinite order in E_1, so the group of rational points $E_1(\mathbb{Q})$ is infinite. Second, $\mathscr{C}$ covers the elliptic curve

$$E_2 : Y^2 = X^3 h(1/X) \,.$$

Any rational point (t, y) on $\mathscr{C}$ such that $t \neq 0$ yields a finite rational point

$$(X, Y) = (1/t^2, y/t^3) \in E_2(\mathbb{Q}) \,.$$

A calculation in `Magma` shows that $E_2(\mathbb{Q})$ is trivial, consisting only of the point at infinity. Hence $\mathscr{C}$ has no finite point (t, y) with $t \neq 0$, and $f - t$ never factors completely over $\mathbb{Q}$ for $t \in \mathbb{Q} \setminus \{0\}$.

Professor Elmer Rees CBE (1941–2019)

Ed Crane writes: Rubel's problem was the last mathematical question that I discussed with my friend Elmer Rees, whom I got to know when he was the first director of the Heilbronn Institute for Mathematical Research. Although he was an algebraic topologist, Elmer had worked before on problems about the factorization of polynomials over the integers. In Oxford in the 1970s, he wrote a brief note with Walter Feit, *A criterion for a polynomial to factor completely over the integers*. They proved this criterion in order to establish a result about the algebraic topology of complex algebraic varieties. Later, in Edinburgh, Elmer and Chris Smyth proved some necessary divisibility properties of solutions of the Prouhet-Tarry-Escott problem [14]. Even though Elmer was already unwell when we talked about Rubel's problem, it sparked his characteristic enthusiasm.

Another solution of `Rubel`(7) is

$$f = (x-3)(x-2)(x-1)x(x+1)(x+2)(x+3) \,.$$

To prove this one can use the method of Victor Flynn and Joe Wetherell [8] to enumerate the rational points on the bi-elliptic curve $y^2 = \mathrm{disc}(f - t)$. Their method is elementary, using ideas from the proof of the Mordell-Weil theorem, but we omit the details.

(L-r) Gerd Faltings, Robert F. Coleman. Photo credit: MFO

The Chabauty method

To give an explicit solution to `Rubel`(6), we will use Chabauty's method to enumerate the rational points on a smooth affine curve of the form $y^2 = \mathrm{disc}(f - t)$, where $\deg(f) = 6$. When f has no repeated critical value, this curve has an embedding defined over $\mathbb{Q}$ into a smooth projective curve in $\mathbb{P}^3$.

Chabauty's method originated as a proof of the Mordell conjecture for a restricted class of curves [7], over 40 years before the work of Faltings. Chabauty's method was made effective in the 1980s through Robert F. Coleman's work on p-adic integration.

The Chabauty–Coleman method is a technique for bounding the number of rational points on a smooth curve $\mathscr{C}$ by embedding them in the p-adic points on the Jacobian variety $\mathcal{J} = \mathrm{Jac}(\mathscr{C})$. The *Jacobian* $\mathcal{J} = \mathrm{Pic}^0(\mathscr{C})$ is an abelian variety of dimension g, where g is the genus of $\mathscr{C}$. The *Abel-Jacobi map* is a rational embedding $u_{P_0} : \mathscr{C} \to \mathcal{J}$ defined with respect to a base point P_0. A point P on $\mathscr{C}$ is sent to the divisor class $u_{P_0}(P) = [P - P_0]$.

If p is a prime, base change gives an embedding $\mathcal{J}(\mathbb{Q}) \to \mathcal{J}(\mathbb{Q}_p)$. The p-adic closure of $\mathcal{J}(\mathbb{Q})$ in $\mathcal{J}(\mathbb{Q}_p)$ is a p-adic submanifold. Its dimension $r' = \dim \mathcal{J}(\mathbb{Q}_p)$ is always bounded above by the rank $r = \mathrm{rk}\,\mathcal{J}(\mathbb{Q})$ of the rational points of the Jacobian as an abelian group.

By composing with the Abel-Jacobi map, we have an embedding $\mathscr{C}(\mathbb{Q}) \to \mathcal{J}(Q_p)$. When $r' < g$, Chabauty used the properties of this embedding to show there are only finitely many points on the curve.

In fact, an explicit bound may be given, as was shown by Robert F. Coleman. The proof and full statement of Coleman's theorem requires a type of p-adic integration now called *Coleman integration*, that treats degree 0 divisors as paths of integration and allows one to extend the definition of p-adic integration outside the radius of convergence of the standard definition. We state Coleman's theorem for easy reference (Theorem 5.3 in [13]); see [13] and the references therein for the definition of the Coleman integral and other notation and terms.

Theorem (Coleman). *Let $\mathscr{C}$ be a curve of genus $g \geq 2$ over $\mathbb{Q}$, and let $\mathscr{J} = \mathrm{Jac}(\mathscr{C})$. Let p be a prime of good reduction for $\mathscr{C}$, let $r = \mathrm{rk}\,\mathscr{J}(\mathbb{Q})$, and let $r' \leq r$ be the dimension of the closure of $\mathscr{J}(\mathbb{Q})$ in $\mathscr{J}(\mathbb{Q}_p)$ as a p-adic manifold. Assume that $r' < g$.*

(a) Let ω be a nonzero 1-form in $H^0(\mathscr{C}_{\mathbb{Q}_p}, \Omega^1)$ with the property that, if $Q_i, Q_i' \in \mathscr{C}(\mathbb{Q}_p)$ such that $\left[\sum_i (Q_i' - Q_i)\right] \in \overline{J(\mathbb{Q})}$, then $\sum_i \int_{Q_i}^{Q_i'} \omega = 0$. Such an ω necessarily exists, and we may assume by scaling that it reduces to a nonzero 1-form $\overline{\omega} \in H^0(\mathscr{C}_{\mathbb{F}_p}, \Omega^1)$. Suppose $Q \in \mathscr{C}(\mathbb{F}_p)$, and let $m = \mathrm{ord}_Q\,\overline{\omega}$. If $m < p - 2$, then the number of points in $\mathscr{C}(\mathbb{Q})$ reducing to Q is at most $m + 1$.

(b) If $p > 2g$, then $\#\mathscr{C}(\mathbb{Q}) \leq \#\mathscr{C}(\mathbb{F}_p) + (2g - 2)$.

In the case when $r' < g$ (in particular, when $r < g$), part (b) of Coleman's theorem gives an upper bound on the number of rational points on $\mathscr{C}$. Part (a) may be used to give a narrower upper bound in many cases. However, this bound is not always tight.

In practice, it seems that the Chabauty-Coleman method *can* be used — in combination with a technique called the Mordell-Weil sieve — to enumerate with proof all the rational points on a curve when $r < g$. When the algorithm terminates, it yields a tight upper bound, but it has not been proven that it always terminates.

A solution of Rubel(6)

We now show that the following polynomial is a solution to Rubel(6):

$$f = (x-2)(x-1)x(x+1)(x+2)\left(x + \frac{5}{2}\right).$$

Let $\mathscr{C}$ be the curve defined by the equation

$$\begin{aligned} y^2 &= 8^2 \mathrm{disc}(f - t) \\ &= 2985984t^5 + 38231885t^4 - 161118396t^3 \\ &\quad - 811349595t^2 + 1302526656t + 4629441600. \end{aligned}$$

This is a hyperelliptic curve of genus 2, so Chabauty can be applied if the rank of the Jacobian is 0 or 1.

The rank $r = \mathrm{rk}\mathscr{J}(\mathbb{Q})$ of the Jacobian $\mathscr{J} = \mathrm{Jac}(\mathscr{C})$ is found using a 2-descent algorithm, which computes the rank of the 2-part of the Selmer group to give an upper bound, combined with a search for points to give a lower bound. Both are implemented by the Magma function RankBounds. To reduce the runtime of RankBounds, we first compute a minimal Weierstrass model for $\mathscr{C}$, given by the equation

$$\begin{aligned} &y^2 - (x+1)y \\ &= 17915904x^5 - 51347635x^4 - 621566x^3 \\ &\quad + 108253979x^2 - 92802025x + 22173442. \end{aligned}$$

When applied to this model, RankBounds returns a lower bound of 0 and an upper bound of 1.

The Jacobian has a nontrivial rational point $u_\infty(P) = [P - \infty]$ coming from the known point $P = (0, 68040) \in \mathscr{C}(\mathbb{Q})$ coming from the factorisation of f. The torsion group of the Jacobian is computed to be trivial with the Magma function TorsionSubgroup; thus, $u_\infty(P)$ has infinite order. So, $\mathrm{rk}\mathscr{J}(\mathbb{Q}) = 1$.

The Magma function Chabauty is then used to enumerate the rational points on $\mathscr{C}$. This function implements the Mordell-Weil sieve as described by Bruin and Stoll [4] to rule out conjugacy classes modulo various primes until the Coleman bounds are made tight (under the assumption that $\mathrm{rk}\mathscr{J}(\mathbb{Q}) = 1$). For our curve $\mathscr{C}$, Magma does a Mordell-Weil sieve using local information at the primes $\{19, 37, 41\}$ and gives the full set of rational points on $\mathscr{C}$ as

$$\mathscr{C}(\mathbb{Q}) = \{(0, -68040), (0, 68040), \infty\}.$$

Thus, $f - t$ does not factor completely over $\mathbb{Q}$, except when $t = 0$. In fact, this method shows that, for $t \neq 0$, the Galois group of the polynomial $f - t$ is never contained in the alternating group A_6.

A challenge: Can you solve Rubel(4) or Rubel(5)?

Current developments of Chabauty's method

The past few years have seen the development of new variations on Chabauty's method that allows one to go beyond the $r < g$ regime (or even $r' < g$). The far-reaching theory of non-abelian Chabauty developed by Minhyong Kim allows one to replace the Jacobian variety with any one of a collection of non-abelian analogues [11]. Balakrishnan and Dogra have refined a special case of Kim's ideas into an effective method called *quadratic Chabauty*. They have used quadratic Chabauty to enumerate the rational points on specific curves, including hyperelliptic curves with $(g,r) = (2,2)$ [1] and $(g,r) = (2,3)$ [2], and, jointly with Müller, Tuitman, and Vonk, to a non-hyperelliptic curve with $(g,r) = (3,3)$ [3].

Meanwhile, classical Chabauty continues to pay off in a wide array of problems. An application to a problem in fluid dynamics may be found in [12]; Lemma 18 therein provides a fully worked example of the Chabauty-Coleman method.

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FURTHER READING

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