

Elmer Gethin Rees, 1941–2019

by John Jones



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Elmer Rees died on 4 October 2019. He was born on 19 November 1941. He was a Welsh-speaking Welshman and this was very important to him. He was very active in all aspects of mathematical life, research, teaching and service. He was also great fun, very good company and a good friend to many people. He is survived by his wife Mary; they were married in New York in 1970. Elmer and Mary have two sons Gethin and David, born in 1980 and 1982, and two granddaughters Nola and Audrey, born in 2016 and 2018.

1 | EARLY LIFE

Elmer Gethin Rees was born at home on 19 November 1941 in Llandybie, Carmarthenshire, South West Wales, to William Powell Rees and Margaret Jane Davies. His father was a farmer and Elmer grew up on the family farm. He was one of two children, his brother Euros (a Welsh name pronounced aye-ross) was nine years older. By all accounts Elmer enjoyed his youth in rural South West Wales: growing up on the farm, hobnail boots, mucking out the cow shed and all that goes with it. He enjoyed the outdoor lifestyle and loved having a dog. He had an interest in wild flowers from a young age and enjoyed identifying them and photographing them for most of his life. He had to work hard on the farm and he also had a long walk to school.

A favourite story from his youth was about a time when he did not take enough care over the cows. He got into trouble as they escaped into nearby woods and ate wild garlic. The milk they produced tasted of garlic and it could not be sold at the market, after all it would be enough to spoil anyone's tea.

The family, of course, spoke Welsh at home. They also spoke Welsh in his primary school and it was not until Elmer went to secondary school, Llandeilo Grammar School, that he began to learn English. The Welsh Grammar schools of that time were very good schools which concentrated on giving their pupils a sound, solid education and turning out good rugby teams. A small minority of the pupils went on to University and it is fair to say that Oxford and Cambridge were not really on their agenda. Nonetheless, Elmer won a place as an undergraduate at Cambridge. He was the first pupil from Llandeilo Grammar to go to Cambridge. He used to say that as a native Welsh

speaker he did not understand all the words used in the interview, and had to guess their meaning. However, and typically, he clearly found convincing answers.

2 | UNDERGRADUATE AND POSTGRADUATE STUDIES

Elmer was an undergraduate at St Catharine’s College, in Cambridge. He graduated from Cambridge in 1963 and went to Warwick to do a PhD under the supervision of David Epstein. David remembers Elmer as a very independent student, very cheerful and full of energy. On one occasion David was looking for someone to build steps in his garden, Elmer volunteered and did the job, saying that he was a farmer’s son, and farmer’s sons can build anything.

David writes: ‘Elmer was one of my first PhD students, and was among the very first graduates of the University of Warwick. He was awarded a PhD in 1967, but the first Warwick undergraduates to graduate did so only in 1968. I was Elmer’s PhD supervisor, though Elmer soon knew more algebraic topology than I did. I was pleased when he found his own problem and solved it without help from me.’



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This the first photograph of the Warwick Mathematics Department (in 1965). There are 12 people, Elmer is in it, so is David Epstein, and so too is David’s supervisor, Christopher Zeeman. Three mathematical generations in a department of 12 people!

It is clear that Elmer had already made contact with Michael Atiyah by this time. Elmer was given an Honorary Degree from Edinburgh in 2008 and in his response to the Laudation given by Michael, Elmer writes ‘I heard of Sir Michael when I was an undergraduate, he was described as an up and coming UK mathematician; his subsequent achievements have shown how true this assessment was. On a personal level, I attended a seminar series given by him when I was working towards my PhD. I have known him as a colleague and friend for almost forty years.’

After his PhD in Warwick, Elmer held short-term positions at the University of Hull (1967–1969), the Institute for Advanced Study in Princeton (1969–1970) and Swansea University (1970–1971).

3 | OXFORD (1971–1979)

In 1971, Elmer went to Oxford as a Fellow in St Catherine's College. Graeme Segal was also a Fellow in St Catherine's when Elmer joined the College, and in 1973 Michael Atiyah returned to Oxford after three years at the Institute for Advanced Study in Princeton, as a Professorial Fellow in St Catherine's. That is a special trio of pure mathematicians eating lunch together!

A large part of the duties of a College Fellow is, of course, organising and giving tutorials to undergraduates. Elmer was naturally good at this, and his students saw him as approachable and friendly, as well as helpful and down-to-earth.

On a personal note, I went to St Catherine's in 1972 to do a DPhil under Elmer's supervision. Indeed I was Elmer's first PhD student. I too am Welsh and the Welsh people have a particular passion for rugby. I remember very clearly a meeting with Elmer in the October of my first term in Oxford, I think it was the second meeting I had with him. When I got to his room in St Catherine's he was listening to the radio broadcast of the famous rugby game when Llanelli (now known as the Scarlets), who Elmer supported all his life, beat the New Zealand All Blacks. Mathematics was completely abandoned in favour of rugby. It was then that I realised that I was really going to get on very well with my supervisor.

At that time the geometry and topology group in Oxford was large and very active (as it still is). My count is ten permanent staff plus a collection of research fellows and research students and a steady flow of distinguished visiting mathematicians. It also had a very informal and inclusive atmosphere. On Mondays in term time there was Michael Atiyah's Geometry and Analysis Seminar at 3 pm and Ioan James's Topology Seminar at 5 pm. The hour between the two seminars was spent in the tea room discussing mathematics and writing on the tables. Clearly some explanation is required at this point. The surface of the tables in the tea room was the same as that of whiteboards and it was conventional to use them as such when talking about mathematics over tea. Finally, the truly hardy, often led by Wilson Sutherland, went to the Lamb and Flag pub just down St Giles from the old Mathematics Institute to finish off the day with a drink. Elmer absolutely thrived in this environment.

Elmer fully understood the importance of international collaboration, conferences and travel for mathematics and mathematicians. This is one of the very important themes running through his career. During his time in Oxford he took full advantage of the opportunities to travel. He had visiting positions in Berkeley (October–December 1973), Sydney (Summer 1974), Mexico City (Summer 1975), IHES (April–July 1977). These visits were very important for him: for example, his time in Berkeley initiated his extensive collaboration with Emery Thomas.

He was also willing to do the organisational work needed to make conferences and research workshops happen, and he did it very well. For example, he organised the Third Oxford Topology Symposium in 1978. This was the first of three large linked international conferences he organised in the space of seven years.

Elmer gave an undergraduate lecture course on geometry in Oxford which is the basis of his book, *Notes on Geometry* [21]. His interest in geometry was stimulated by Christopher Zeeman while he was a graduate student in Warwick, and as Elmer says in the introduction he was strongly encouraged by Michael Atiyah and Graeme Segal to give these lectures. The purpose of the book, to quote from the preface, is 'to fill a rather obvious gap' in the undergraduate literature. I was still

in Oxford, by this time as a research fellow, when he gave these lectures and I went to them as this was certainly a gap in my undergraduate education. It was interesting to see the number of PhD students in the audience. There really was a gap in the literature, and the book fills it beautifully.

4 | EDINBURGH (1979–2005)

Elmer was appointed to the Chair of Mathematics in Edinburgh in 1979. This is one of the four original Chairs dating from 1583. With typical wit he commented ‘they would do anything to avoid appointing a Sassenach’. For some time the primary focus of research activity in mathematics in Edinburgh had been in analysis. Elmer’s interests in geometry and topology brought a new research direction to the department and naturally attracted PhD students, younger staff and visitors.

John Ball, who was at Heriot-Watt when Elmer arrived in Edinburgh, writes as follows.

‘At the time I was at Heriot-Watt University, and we felt somewhat looked down upon by the older University of Edinburgh. This all changed with Elmer’s arrival, and he immediately went out of his way to build bridges between the two mathematics departments. The resulting close collaboration led first to the founding of the International Centre for Mathematical Sciences (ICMS), initially located in the birthplace of James Clerk Maxwell and now a highly successful research centre known throughout the world, and later to the creation of the Maxwell Institute for Mathematical Sciences, a unique research partnership between the two universities. Without Elmer this would never have happened.’

The collaboration between Edinburgh and Heriot-Watt that John Ball is talking about established an outward-facing international research centre, ICMS, and an internal (to Edinburgh) research partnership, the Maxwell Institute. Both Institutes have developed over time and a straightforward internet search will give an impression of their current size and level of activity. In their present form, they really do help make Edinburgh one of the major UK centres for the Mathematical Sciences.

One of the first things Elmer did in Edinburgh was to organise a big international conference in 1981 which he called the Fourth Topology Symposium (Edinburgh), a neat way to describe its relation to the Third Oxford Topology Symposium which he organised in Oxford in 1978. This was followed by a Durham Symposium in Homotopy Theory in 1985 which I helped him organise. The period covered by these three conferences, 1978–1985, saw some spectacular advances in algebraic topology. For example, pseudo-isotopy theory and Waldhausen K-theory were really starting to have an impact; the Segal conjecture and the Sullivan conjecture had been proved; and there were some very deep new results in global stable homotopy theory, concerning localisation and periodicity. All these topics are still very important in algebraic topology. The proceedings of the Durham Symposium are published in the LMS Lecture Note Series [12] and contain several survey articles which give a very clear account of how these things stood at the time of the Durham Symposium.

I remember an interesting moment in the Durham Symposium. Mike Hopkins was giving a lecture on the great work he, Ethan Devinatz, and Jeff Smith were doing to solve conjectures due to Doug Ravenel. Frank Adams was in the audience and he was, of course, very interested and excited by the work. He wanted to ask Mike a lot of questions, perhaps too many! Elmer turned to Frank and asked politely if he would let Mike get on with his lecture. Frank, who was very meticulous about such matters, took the point and saved the rest of his questions until after the lecture.

In his time at Edinburgh, Elmer took on a lot of administrative and organisational work both in the University and in the mathematical community. Internally he took on the usual work expected of a well-respected senior professor. For example, he was Head of Department from 1983 to 1986. He was on the Dean's Committee in Edinburgh for 10 years, he was involved in various University management committees, and also promotions and appointments committees. He was on the ICMS Executive Committee in 1989–2002. Externally he served the London Mathematical Society in many capacities, for example two terms on the LMS Council, and as Vice President in 1994–1996. He was also Vice President of the Edinburgh Mathematical Society in 1989–1990 and 1991–1993, and President in 1990–1991. He did a lot of senior editorial work for both the LMS and the Royal Society of Edinburgh, and he was on the Council of the RSE in 1987–1989. He was on the Isaac Newton Institute Scientific Steering Committee from 1999 to 2005, and was Chair of this committee for the last 3 years of his term. Despite taking on this administrative load it was a very productive time for both his research and PhD students, as his publications and the list of his PhD students show.

During this period Elmer developed a long-term collaboration and friendship with Victor Buchstaber (from Moscow), which lasted well over 25 years. Both of them were at a conference in Aberdeen in 1991, and after the conference was over, Victor stayed in Elmer's home in Edinburgh for a couple of days and they spent a lot of time discussing mathematics. From this time on, Victor became a regular visitor to Edinburgh funded by various grants and visiting fellowships. In turn, Elmer visited Victor in Moscow on many occasions. One of Elmer's particularly notable visits to Moscow was the occasion when he gave a plenary talk, on his joint work with Victor, in the conference 'Kolmogorov and Contemporary Mathematics' held in Moscow in June 2003 to celebrate Kolmogorov's centenary. The paper [9] is based on this talk.

From 1997, Michael Atiyah was an Honorary Professor in Edinburgh, after his retirement from Cambridge. Elmer, Michael and Victor interacted very well, and Victor has many happy memories of their very lively mathematical (and other) discussions.

We Welsh people always return at some point or other to rugby. Shortly after Elmer went to Edinburgh, I got an invitation to give a seminar on a Friday, which happened to be the day before the Wales–Scotland rugby match in Edinburgh. Elmer wrote in his letter of invitation 'of course I've got tickets for the match'. This 'rugby seminar', as it was christened by Andrew Ranicki, continued as long as Elmer could get tickets. What a way to spend a weekend, mathematics and rugby in the company of Elmer.

5 | ICMS

Elmer's most important contribution to mathematics in Edinburgh, and probably to the UK Mathematical community, is the creation of the ICMS (International Centre for Mathematical Sciences). This was jointly established by the Edinburgh and Heriot-Watt universities in 1990.

It is very interesting to look back at those times. In April 1989, Abdus Salaam was presented with the first Edinburgh Medal and in his acceptance speech he challenged Edinburgh and Scotland to create research institutes modelled on ICTP (International Centre for Theoretical Physics) in Trieste. Salaam was the founder of ICTP and also the director at that time. Two months later, and independently of course, after discussions with the LMS and others, the research funding agency SERC (now called EPSRC) announced a competition to establish a national mathematics research centre in the United Kingdom.

This all led to a joint proposal from Edinburgh and Heriot-Watt, written by Elmer and John Ball, of course with help from many other people, for such a research centre in Edinburgh. In November 1989, the national centre was awarded to Cambridge and this is the origin of the Isaac Newton Institute for Mathematical Sciences. The joint Edinburgh Heriot-Watt bid was well-praised. But for Edinburgh, the most important thing about the whole process was the momentum it generated for establishing a mathematics research centre in the city.

With funding from the Scottish regional development agency, and the support of the Edinburgh City Council, a feasibility study into the establishment of a mathematics research centre in Edinburgh was set up. A protocol agreement was signed in December 1989 which, in effect, committed those who signed it to work together to establish this mathematics research centre, providing, of course, the feasibility report was sufficiently positive. The feasibility study did indeed support the proposal. The next step was to establish an Executive Committee chaired by John Ball, with Elmer as Vice Chair, and at the first meeting of the Executive Committee in May 1990 the name 'International Centre for Mathematical Sciences' was agreed. Elmer chaired this Executive Committee from 1993 to 1996, and then became Vice Chair from 1996 to 1999.

The first mathematical event, Symposium on Geometry and Physics, took place in March 1991, organised by Elmer. There is an ICMS image and video gallery in the web-based service Flickr which can be accessed from the ICMS website. The conference photograph and many other photographs can be found on the website. This archive of photographs is very interesting for two reasons: it gives a clear impression of the amount and level of activity ICTP has generated; and it is great to see old photographs of mathematicians.

The signing of the protocol agreement in December 1989 was a very important event in the establishment of ICMS. It was also very important for another reason. It took place in the City Chambers, it was televised, and in the broadcast interviews Elmer mentioned that the institute was still looking for premises. This was picked up by the Scottish Director of the James Clerk Maxwell Foundation who subsequently offered the use of 14 India Street, Maxwell's birthplace. This led to ICMS renting permanent accommodation there from 1994. This is usually attributed to good fortune but, to coin an old-fashioned headmaster's phrase, the only place where good fortune comes before hard work is in the dictionary.

Angus Macintyre became the first director of ICMS in 1993 and with the move into permanent accommodation in 1994 the ICMS was indisputably up and running. Now, ICMS is a very well-established international research centre. Through the work of many people over the years, it has a full-time permanent director and premises. It has a sustainable funding stream and runs a large and varied programme of international research activities. But go back to John Ball's comment: 'Without Elmer this would never have happened.'

6 | HEILBRONN INSTITUTE (2005–2009)

Elmer left Edinburgh in 2005 to become the first director of the Heilbronn Institute for Mathematical Research (HIMR). HIMR is a partnership between GCHQ (Government Communications Headquarters) in Cheltenham and the mathematical community, and its working model is that members spend half their time pursuing classified research in support of GCHQ and the other half doing their own personal research.

GCHQ had been using academic consultants (also known as secondees) successfully for some long while, but it had become clear it was time for a change. Mathematical techniques were becoming more and more important in the classified research of GCHQ and something new was

needed to attract the increasingly high level of mathematical expertise, together with the team-working required for classified research. It was decided to form a new institute, a kind of modern version of the war-time Bletchley Park. There would be a 3-year trial period and all contracts were designed so that the institute could be shut down with 3 years' notice.

So, HIMR was founded in 2005, Elmer was appointed as Director, Richard Pinch as Deputy Director and two administrative staff were also appointed. But there were no members of the institute. Elmer's immediate tasks as Director were: (a) to populate the institute with members; (b) to establish the internal ethos of the institute; (c) to raise the external profile of the institute. Of course, these three tasks interact in a very significant way.

To put the first task in a rather stark way: how exactly do you recruit people to a job when you cannot tell them anything about half of the job, but you do have to tell them that the institute could be closed down with 3 years' notice, and they would have to sign the Official Secrets Act? Elmer had the honesty to set things out clearly and succinctly. Most important of all, he had the credibility in the mathematical community to be seen as someone who could make a success of a venture like this.

He organised conferences and seminars to raise the profile of HIMR in the mathematical community and attract people to visit the institute. Naturally, he used these as opportunities to talk to individuals about the possibility of becoming a member of HIMR in some appropriate capacity. When Elmer stepped down in 2009 there were something like 30 members, with a ratio of roughly three research fellows to one academic consultant. He sat on all appointment panels.

One thing to reflect on is that for this period of time Elmer was a civil servant and he had to interact in a serious way with the Directorate at GCHQ. He also had to maintain his credibility in the mathematical community. There are real cultural differences between these two groups and it takes something very special to pull this off. Clearly Elmer succeeded: for example, he managed to persuade GCHQ that in order to attract quality mathematicians to HIMR there should be sensible and generous leave arrangements. But his methods might well be classified information for some time to come.

The first review in 2005–2006 contains some glowing praise, but I will restrict myself to one quote: 'The future potential of the institute is rich. The first year has laid down the foundations of an active and effective research establishment.' HIMR was made permanent in 2011, 2 years after Elmer stepped down as Director.

If we look now at HIMR it is clearly here to stay. It has expanded to both London and Manchester and it has attracted a lot more funding. It has probably doubled its membership since 2009 but it seems to have maintained the relaxed informal atmosphere of an academic department.

Elmer continued to work with Victor Buchstaber during his time at HIMR and afterwards at Bristol. So, Victor became a regular visitor to Bristol. When Elmer and Victor got together and they had time at the weekends, they would often go off on excursions to see something or visit something. From Bristol they went to Wales and Elmer took the opportunity to show Victor the rural part of South West Wales where he grew up and also to visit Elmer's brother. Victor certainly enjoyed this and I am sure Elmer did as well.

7 | BRISTOL (2009–2019)

After Elmer stepped down from HIMR, he became an Honorary Visiting Professor at Bristol University. He continued to interact with the mathematical community in Bristol, going to seminars and the like. He also continued to work as a part-time consultant on the classified research of

HIMR, and he published a paper [20]. But his health was failing and he passed away peacefully after a long period of ill health.

8 | ELMER'S SPECIAL CONTRIBUTION TO THE MATHEMATICAL COMMUNITY

Elmer made many contributions to mathematics as his lists of papers and PhD students show. What is more, he was willing to take on serious administrative and organisational roles within the UK Mathematics community. For example, both ICMS and HIMR owe a huge debt of gratitude to Elmer for his influence on their formative years. Listing these things makes an impressive CV. He had an extraordinary combination of honesty, underlying seriousness of purpose, a great sense of humour and a seemingly endless collection of anecdotes. This made him very easy, and great fun, to work with.

Elmer's sons Gethin and David commented that in the messages of condolence they received, there were so many references to how good a mentor Elmer was, working with younger colleagues in particular. They also tell a great story which I cannot resist repeating: 'Ida Ranicki described Dad as the most charming mathematician that she knows. This is a great comment (a) because she didn't mention whether her husband Andrew, who was there, was the second most charming, and (b) not all Dad's jokes were funny, so we are not quite sure what this says about mathematicians.'

But there was a great deal more to his contribution to the mathematical community. I cannot possibly do better than quote John Ball from his eulogy to Elmer at his funeral.

'Elmer had human qualities that made him such a treasured member of the mathematical community and a friend to so many. He was perceptive, tenacious, generous and great company. He was never short of words, and phone calls could last an hour or more. He had opinions on everything and everyone. His fine judgement and collegiality made him a natural choice for committees and delicate assignments. A gossip of the highest class, it was seemingly impossible to tell him something which he didn't already know. He must somehow have had to curb these instincts when he became the founding director of the Heilbronn Institute, which is funded by GCHQ, in Bristol. As Director he set its tone and intellectual ethos. For his retirement conference in 2009, Jon Keating had a mug designed with an inscription in Welsh. He approached two Welsh speakers for a translation of "To mark the end of Elmer's reign", and they came back with completely different versions. Elmer said that the one that was used would do, but that he would have come up with something more poetic!'

9 | MATHEMATICS

Elmer's research interests were in geometry and topology and his publications cover the full range of this broad field. Scanning his list of papers one immediately sees the breadth of his interests. Looking in more detail at his papers you will see that he was always motivated by a concrete, usually geometric, problem and was willing to use and adapt whatever techniques were needed and relevant. He had a characteristically clear and concise style of writing. He seemed to be able to tell you exactly what you wanted to know with the minimum amount of fuss. Mathematical discussions with him were exactly the same, except that there was plenty of banter, jokes and stories to keep everyone sharp.

I have chosen to give an impression of Elmer's mathematics by focussing on his two main collaborations. The first is with Emery Thomas which began when Elmer was in Oxford, and the second is with Victor Buchstaber which dated from his time in Edinburgh. I have also chosen to comment on a couple of other pieces of Elmer's work but not in any kind of systematic way. But we must begin at the beginning.

9.1 | Thesis

Elmer's thesis was about calculations of K-groups of projective space and applications to two problems. The first is to the study of maps $f : S^n \rightarrow S^{n-k}$ between spheres such that $f(x) = f(-x)$. The second problem concerns multiplication maps on the real projective spaces $\mathbb{RP}^n$. It follows from Adams's famous work on the Hopf invariant that such a multiplication map exists if and only if $n = 1, 3, 7$. In the case $n = 1$ it is easy to check that there is only one such multiplication map, up to the appropriate notion of homotopy. It was known that when $n = 3$ there are 768 such maps up to homotopy and Elmer gave a very elementary proof of this fact. He then went on to show that in the case $n = 7$ there are 30,720 such maps up to homotopy. His thesis led to three papers [13–15].

9.2 | Rees and Thomas

The papers [23–27], written in collaboration with Emery Thomas, are concerned with applying complex cobordism to the study of the topology of complex algebraic varieties. The papers [3, 11, 18, 19] are also closely related to this general theme.

René Thom's theory of complex cobordism involves closed (compact with no boundary) smooth k -dimensional manifolds M^k embedded in $\mathbb{R}^{k+2n}$, with a complex structure on the normal bundle of the embedding. Thom constructs a space $\text{MU}(n)$ such that the cobordism classes of such embeddings are the same as the homotopy groups $\pi_{k+2n}(\text{MU}(n))$. If $k < 2n$ then these groups are independent of n . This is the stable case and the group $\pi_{k+2n}(\text{MU}(n))$ with $k < 2n$ is known as the group MU_k of complex cobordism classes of k -dimensional manifolds.

One interesting feature of this work is that many of the best results use the unstable cobordism groups $\pi_k(\text{MU}(n))$ where $k \geq 2n$. Most of the technical work needed to calculate some of these groups is done in [25] and the relevant calculations with Chern classes are carried out in [23]. These techniques have many applications.

9.2.1 | Isolated singularities

Let V be a complex affine algebraic variety in $\mathbb{C}^{n+k}$, of complex dimension k with an isolated singular point p . Is it possible to deform V a little bit so that it becomes smooth? Thom, in unpublished work, had constructed an example where no such deformation exists. The link of an isolated singularity p is the intersection of V with a (sufficiently small) sphere with centre p . It is a smooth manifold, as the singularity is isolated. If there is such a deformation then this link is the boundary of a complex manifold. So, Thom constructs an example where the link does not have this property.

In [25], Elmer and Emery refine and generalise Thom's method to give a geometric construction which yields an invariant of an isolated singularity in the cobordism group $\pi_{2n+2k-1}(\text{MU}(n))$. This

invariant must vanish if such a deformation exists. If $n > k$ (the stable range) this group vanishes. But if $n \leq k$ (the unstable range) it can, and it often is, non-zero. They now apply their techniques to compute these invariants.

Some of the most interesting examples are ‘cone’ singularities. Cone singularities are obtained by starting with a non-singular complex k -dimensional variety in $\mathbb{CP}^{n+k-1}$. The corresponding affine variety in $\mathbb{C}^{n+k}$ has an isolated singularity at the origin. One particularly interesting example is given by the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^{n-2}$ in $\mathbb{CP}^{n-1}$. In this case they show the invariant of this particular cone singularity is a generator of the unstable complex cobordism group $\pi_{4n-1}(\text{MU}(n))$.

9.2.2 | Intersections of complex subvarieties

Let V be a complex projective variety in $\mathbb{CP}^n$. When are there compact projective varieties $X, Y \subseteq \mathbb{CP}^n$ such that X and Y intersect transversally and $V = X \cap Y$? Again there is a stable/unstable aspect to this problem. If X and Y , of complex dimension k and l respectively, intersect transversally then their intersection V has complex dimension $k + l - n$. In particular, it is empty in the stable case $n \geq k + l$.

Elmer and Emery use the complex cobordism ring $\text{MU}^*(\mathbb{CP}^n)$ to study this problem. One of the most important parts of complex cobordism theory is that to any reasonable topological space X it assigns a graded ring $\text{MU}^*(X)$ which is, in technical terms, a generalised cohomology theory of X . The subvarieties X, Y, V of $\mathbb{CP}^n$ define elements $[X], [Y], [V]$ in $\text{MU}^*(\mathbb{CP}^n)$ and assuming that V is the transversal intersection of X and Y it follows that $[V] = [X][Y]$ where the product is in the ring $\text{MU}^*(\mathbb{CP}^n)$.

In [26], Elmer and Emery use their calculations of unstable cobordism groups to give relations between invariants of V, X and Y which must hold if V is the transverse intersection of X and Y . This is not really the place to try to summarise the details of these relations. However, the applications are of two kinds. The first is to prove that V cannot be the transverse intersection of X and Y . The second is, assuming that V is indeed the transverse intersection of X , it gives a way of calculating the algebraic invariants of V in terms of those of X and Y . The final sections of the paper [26] give plenty of examples to illustrate both the methods and the results.

9.2.3 | Realising homology classes by complex normal submanifolds

Let V be a closed (that is compact with no boundary) oriented manifold. A closed submanifold X of M is a complex normal submanifold if the normal bundle of X in V comes equipped with a complex structure. Elmer and Emery look for necessary and sufficient conditions for a homology class in $H_r(V)$ to be represented by a complex normal submanifold of V . Any complex submanifold of a complex manifold is complex normal but the notion of a complex normal submanifold is weaker than the notion of a complex submanifold.

The specific case Elmer and Emery study in [27] is where V is $4n$ -dimensional and $u \in H_{2n}(V)$. First they define an integer invariant $\gamma(V)$ which depends only on V . Now given $u \in H_{2n}(V)$ they define an integer $\Delta(u)$ which depends on u . They show that u can be represented by a complex normal submanifold of V if and only if

$$\Delta(u) = 0 \pmod{\gamma(V)}.$$

The proof uses the complex cobordism ring $MU^*(V)$. There are natural homomorphisms $\Gamma_j : MU^*(V) \rightarrow H^*(V)$ $j = 0, 1, \dots, n$ which raises the degree by $2j$. The first of these is given by the Thom class in $H^{2n}(MU(n))$: in general Γ_j is determined by the cohomology class in $H^{2n+2j}(MU(n))$ which corresponds to the Chern class c_j under the Thom isomorphism.

It follows easily from the definition of Γ_0 that if a homology class u in V can be represented by a complex normal submanifold of V then the Poincaré dual $D(u)$ of u in the cohomology of V is in the image of Γ_0 . Elmer and Emery go on to compute the obstructions to $D(u)$ being in the image of Γ_0 . They show that this obstruction lies in a finite cyclic group, $\gamma(V)$ is the order of this group, and $\Delta(u)$ is an element in this group. It is important to note that $\Delta(u)$ is not a linear function of $u \in H_{2n}(V)$: it has a quadratic term. So, the theorem they prove is that $D(u)$ is in the image of Γ_0 if and only $\Delta(u)$ is zero in the obstruction group.

They give the following, very striking, application of this result. Let T_1, T_2 be complex tori of complex dimension n and let $T = T_1 \times T_2$. They define an integer $\rho(n)$ with the following property. The homology class of $r_1[T_1] + r_2[T_2]$ in T can be represented by a complex normal submanifold if and only if

$$r_1 r_2 = 0 \pmod{2^{\rho(n)-1}}.$$

So, if $\rho(n) > 1$ then, even though T_1, T_2 are complex submanifolds of T , the homology class $[T_1] + [T_2]$ cannot even be represented by a complex normal submanifold of T .

The integer $\rho(n)$ is defined as follows. Let m be an integer and let $\alpha(m)$ be the number of ones in the dyadic expansion of m . So, $\alpha(2^r) = 1$ and $\alpha(2^r - 1) = r - 1$. Now let $\rho(n)$ be the smallest positive integer k such that $\alpha(n + k) \leq 2 \leq \alpha(n + 2k)$. So, $\rho(n) \geq 1$ and $\rho(n) = 1$ if and only if $n = 2^a + 2^b - 1$ where $a, b \geq 0$ are integers. The first time $\rho(n) = 2$ is when $n = 6$.

9.3 | Buchstaber and Rees

One of the main general themes in the collaboration between Elmer and Victor Buchstaber is the use of multivalued functions. Let X be a space and let $SP^n(X)$ be the n -fold symmetric product of X . Explicitly $SP^n(X) = X^n / \Sigma_n$ is the quotient space of the action of the symmetric group Σ_n on the n -fold product given by permuting coordinates. Sometimes the term n -set is used for an element of $SP^n(X)$; it is a subset of X with multiplicities and total cardinality n . A multi-valued continuous function from Y to X is, by definition, a continuous function $Y \rightarrow SP^n(X)$. Multivalued functions arise in geometry and topology in many different ways. One general way is in the theory of covering spaces. One of the most studied examples is the multivalued function which assigns to a complex polynomial its roots in $\mathbb{C}$. This multivalued function leads to a homeomorphism $\mathbb{C}P^n \rightarrow SP^n(\mathbb{C}P^1)$.

9.3.1 | On n -valued groups

The papers [4–6] introduce, and study in detail, the concept of an n -valued group. An n -valued group is a space X together with a map $\mu : X \times X \rightarrow SP^n(X)$ satisfying the appropriate notions of associativity, units and inverses. I will give the associativity condition explicitly but not the other two.

Given an n -set $[x_1, \dots, x_n]$ in $\text{SP}^n(X)$ and another point $x \in X$ then we can form two n -sets

$$[(x, x_1), \dots, (x, x_n)], \quad [(x_1, x), \dots, (x_n, x)] \in \text{SP}^n(X \times X).$$

Use the notation $x \times [x_1, \dots, x_n]$ and $[x_1, \dots, x_n] \times x$ for these two n -sets. Then the associativity condition on a map $\mu : X \times X \rightarrow \text{SP}^n(X)$ is

$$\mu(x \times \mu(y, z)) = \mu(\mu(x, y) \times z)$$

These n -valued groups are related to the notion of a hypergroup (as described in a survey article by Livitnov cited in [4]). They are also related to the notion of n -valued formal groups in algebraic topology as studied by Buchstaber and Novikov. The emphasis on the n -valued groups allows a more detailed analysis. Combined with a geometric/topological point of view it also leads to a strikingly rich collection of examples.

A general class of examples are the n -coset groups. Let G be group and let A be a finite group of automorphisms of G . Then the quotient $X = G/A$ has a natural n -valued product where n is the order of A . Explicitly, let $\pi : G \rightarrow X$ be the quotient map and $x \in X$. Given $x \in X$, then by $\pi^{-1}(x)$ we mean the n -set in $\text{Sp}^n(G)$ consisting of the points $g \in G$ such that $\pi(g) = x$. Now define the map $\mu : X \times X \rightarrow \text{Sp}^n(X)$ by $\mu(x, y) = \pi(m(\pi^{-1}(x), \pi^{-1}(y)))$ where $m : G \times G \rightarrow G$ is the multiplication in G .

They go on to describe multivalued group structures on euclidean spaces and spheres. One particular such example, that of a 2-valued group structure on the 3-sphere S^3 was very important for them at the very beginning of their work. Two very nice examples are given by using the Weierstrass elliptic function $\wp$ and the Jacobi elliptic function sn to define two different 2-group structures on $\mathbb{CP}^1$. Further examples also involve dynamical systems.

9.3.2 | On n -valued Hopf algebras

In [4–7], Elmer and Victor introduce the purely algebraic notion of a n -Hopf algebra. Suppose G is a finite discrete group. Let $C(G)$ be the ring of complex valued functions on G . The multiplication map $G \times G \rightarrow G$ defines a coproduct $C(G) \rightarrow C(G) \otimes C(G)$ and the product and the coproduct define a Hopf algebra structure on $C(G)$. If G is a finite n -group the ring $C(G)$ also has a coproduct but this time the relation between the product and the coproduct is not the relation which defines a Hopf algebra. Axiomatising the relation between the product and the coproduct on $C(G)$ where G is an n -group gives the definition of an n -Hopf algebra. This then allows them to study n -groups by using their associated n -Hopf algebras.

They use n -Hopf algebras to show that not every n -group is an n -coset group. In the case of a topological n -group its cohomology is an n -Hopf algebra and they use this to show that there are connected spaces which do not admit any continuous n -group structure.

Here is another striking example. Let G be a compact Lie group. Then the cohomology of BG , the classifying space of G is, of course, a ring but it does not have a natural Hopf algebra structure. In [7] they show that the cohomology of BG does have a natural n -Hopf algebra structure where n is the order of the Weyl group of G .

9.3.3 | Functions on a symmetric product

In [8], Elmer and Victor inevitably their work on n -groups leads Elmer and Victor to think about the functions on a symmetric product $\mathrm{SP}^n(X)$. Let X be a compact Hausdorff space and let $C(X)$ be the ring of complex-valued functions on X , suitably topologised. Then the evaluation map

$$\mathcal{E} : X \rightarrow \mathrm{Hom}(C(X), \mathbb{C}) \quad \mathcal{E}(x)(f) = f(x)$$

is a homeomorphism onto the subspace of all ring homomorphisms from $C(X) \rightarrow \mathbb{C}$. This is a famous result of Gelfand and Kolmogorov.

Elmer and Victor prove the corresponding result for the evaluation map

$$\mathcal{E} : \mathrm{SP}^n(X) \rightarrow \mathrm{Hom}(C(\mathrm{SP}^n(X)), \mathbb{C})$$

$$\mathcal{E}[x_1, \dots, x_n](f) = f(x_1) + \dots + f(x_n).$$

For this, they introduce Frobenius n -homomorphisms. This notion arises in work of Frobenius in the late 1890's. Let G be a finite group and A its group algebra. Let ρ be the character of an n -dimensional representation of G . Then Frobenius shows that ρ can be extended to a linear map $\rho : A \rightarrow \mathbb{C}$ such that $\rho(1) = n$ and $\Phi_{n+1}(\rho) = 0$ where $\Phi_{n+1}(\rho)$ is an explicitly constructed linear map $A^{\otimes n+1} \rightarrow \mathbb{C}$. They show in [8] that $\mathcal{E} : \mathrm{SP}^n(X) \rightarrow \mathrm{Hom}(C(\mathrm{SP}^n(X)), \mathbb{C})$ is a homeomorphism onto the subspace of continuous functions f satisfying two conditions: $f(1) = n$ and $\Phi_{n+1}(f) = 0$.

One of the very interesting points about their proof is that it is constructive. Given a function f satisfying the above conditions, they construct a n -set $[x_1, \dots, x_n]$ such that $\mathcal{E}[x_1, \dots, x_n] = f$. The case $n = 1$ gives a constructive proof of the Gelfand–Kolmogorov theorem: given a continuous algebra homomorphism $f : C(X) \rightarrow \mathbb{C}$ it produces a point $x \in X$ such that $f = \mathcal{E}(x)$.

9.4 | Vector bundles

Elmer's early paper [16] does exactly what it says in the title. Let X be a CW complex of dimension $2n$ and let E be a k -dimensional complex vector bundle over X . If $k > n$ then E always has a section. In this context 'has a section' means 'has a never-zero section'. This is the stable case. If $k = n$ then E has a section if and only if the n th Chern class c_n of E vanishes. If it has two sections, (in this context 'has two sections' means 'has two linearly independent sections') then both c_n and c_{n-1} vanish. Elmer gives a condition which ensures that if E is an n -dimensional complex vector bundle over X and $c_n = c_{n-1} = 0$, then E has two sections. He applies this to solve a problem about vector bundles on $\mathbb{CP}^4$ raised by Rolf Schwarzenberger. It is a short and very neat paper.

This leads to the paper [2] on vector bundles over $\mathbb{CP}^3$ written by Elmer and Michael Atiyah. The main theorem is that every continuous 2-dimensional complex bundle on $\mathbb{CP}^3$ has a holomorphic structure. Of course this holomorphic structure is not necessarily unique.

The main difficulty is that continuous 2-dimensional complex vector bundles over $\mathbb{CP}^3$ are not classified by their Chern classes alone; there is another $\mathbb{Z}/2$ -valued invariant α involved. The Chern classes c_1, c_2 of E are necessarily even; this is a topological condition. By this time Horrocks had constructed holomorphic vector bundles over $\mathbb{CP}^3$ with arbitrary even c_1 and c_2 . So, the strategy is to compute $\alpha(E)$ where E is one of the bundles constructed by Horrocks, in terms of the

parameters in the construction. This then leads to their theorem. Computing this $\mathbb{Z}/2$ invariant is a delicate matter.

Now look at [17]. Here Elmer applies his expertise with vector bundles over complex projective spaces to answer a question of Schneider. As you can see from the title it fits in perfectly with Elmer's paper [16]. Finally, look at [1] and you will find the 'Rees bundles', that is the bundles constructed in [17], playing a prominent role.

9.5 | Two of the outliers

The paper [10] with Walter Feit gives a nice argument in number theory to give a criterion for a polynomial to factor completely over the integers (as it says in the title).

The paper [22] with Chris Smyth is on the Tarry–Escott problem. This is a classical problem in number theory: find sequences of integers $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$ such that

$$a_0^j + a_1^j + \dots + a_n^j = b_0^j + b_1^j + \dots + b_n^j$$

for $j = 1, \dots, n$ but not for $j = n + 1$.

It is interesting to speculate on the question of why Elmer should be interested in such things. Well, Elmer really was interested in interesting problems wherever they came from and that may well be the answer. However, his work with Emery Thomas uses cobordism theory in an essential way. In cobordism theory calculations with characteristic classes are very important and such calculations are closely related to symmetric functions. If you look up where [10] is used in [23] you will see why Elmer and Emery needed the result in [10]. Now power sums are symmetric functions and this gives a link with the Tarry–Escott problem.

10 | HONOURS

Fellow of the Royal Society of Edinburgh (FRSE), Honorary Degree (Edinburgh), Fellow of the Learned Society of Wales (FLSW), Commander of the Order of the British Empire (CBE).

11 | PhD STUDENTS

11.1 | Oxford

- (1) J. D. S. Jones, *A problem in algebraic topology*, 1976.
- (2) N. Brotherton, *Parallelisable four-manifolds not admitting a complex structure*, 1978.
- (3) A. P. Bahri, *Some problems in algebraic topology*, 1980.

This list needs some explanation. In those days, and perhaps these, the process for admission in Oxford to a DPhil (i.e. PhD) required a title for a DPhil thesis. There was a process for changing the title. Neither Tony Bahri (whose original title was much more ambitious than mine) nor I took advantage of this process. Neil Brotherton did and so his original title remains in decent (but predictable) obscurity.

11.2 | Edinburgh

- (4) T. P. Killingback, *Global structure of quantum field theory*, 1984.
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- (16) F. Araujo, *Einstein homogeneous Riemannian fibrations*, 2008.

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